Economic growth and population models: a discrete time analysis

Abstract

This paper studies an extension of the Mankiw-Romer-Weil growth model in discrete time by departing from the standard assumption of constant population growth rate. More concretely, this rate is assumed to be decreasing over time and a general population growth law verifying this characteristic is introduced. In this setup, the model can be represented by a three dimensional dynamical system which admits a unique solution for any initial condition. It is shown that there is a unique nontrivial equilibrium which is a global attractor. In addition, the speed of convergence to the steady state is characterized, showing that in this framework this velocity is lower than in the original model.

Keywords: Mankiw-Romer-Weil economic growth model; discrete time; decreasing population growth rate; speed of convergence.

JEL classification: C62; O41

1 Introduction

An obligatory reference in the studies on economic growth and its determinants, particularly in the empirical ones, is the model developed by Mankiw, Romer and Weil [41]-also known as the Solow model extended with human capital. The model assumes that labour force (associated with the size of the population) grows at a constant rate $n > 0$. This assumption, normally used in the classic growth models (Solow [48], Ramsey [44], Cass [16], Koopmans [38] among others) implies that the population grows exponentially, i.e. in discrete time if the initial population is $L_0$, the population at time $t$ is $L_t = L_0(1 + n)^t$. Assuming that population growing exponentially implies that there is no limit to the size of the population (it tends to infinity as $t$ tends to infinity). This assumption is clearly not sustainable, it do not fits with recent empirical data of the last hundred years [50].

The exponential model conforms the dynamics of the population in initial periods, but it is unable to reflect the fall in the rate of growth due to-for example-lower fertility rate. Verhulst[51] shows that there should be an upper bound for the size of a population.
called load-carrying capacity of the environment, the maximum level of population that an environment can support until it is unable to sustain and feed human activity.

Several experts (Daily[18], Brown[14]) reference that humanity is close to that limit. According to data of the United Nations [50] the rate of population growth has decrease in the last one hundred years and is actually close to 1%. Moreover, the projections for the coming years is that this trend will continue, due to lower rates of fertility. In summary, empirical data reveals two stylized facts: i) the population does not grow at a constant rate, and ii) this rate decreases with time, probably to zero.

Maynard [42] proposes the following properties that characterize a law of population verifying these and other stylized facts:

1. The population grows but is constrained by a maximum size, carrying capacity $L_\infty$:
   \[ L_{t+1} \geq L_t \]

   \[
   \lim_{t \to +\infty} L_t = L_\infty
   \]

2. the growth rate of the population decreases to zero, i.e. $n_t = \frac{L_{t+1} - L_t}{L_t}$ decreases to zero:
   \[ n_{t+1} \leq n_t \text{ and } \lim_{t \to +\infty} n_t = 0 \]

Without loosing generality, let assume that the population law can be represented by an autonomous difference equation, where $L_t$ is the solution of the initial value problem represented by:

\[
\begin{cases} 
L_{t+1} = P(L_t) \\
L_0 > 0 
\end{cases} 
\tag{1}
\]

We assume that function $P(.)$ verifies the following properties:

1. $P(L) \geq L > 0, \forall L \leq L_\infty$.
   This means that the rate of growth of population is non negative: $n(L_t) = \frac{L_{t+1} - L_t}{L_t} = \frac{P(L_t)}{L_t} - 1 \geq 0.1$

2. $\frac{P(L_t)}{L_t} \geq \frac{P(L_{t+1})}{L_{t+1}}$
   That is, the population grows at a decreasing rate: $n(L_t) \geq n(L_{t+1}) \geq 0.2$

\footnote{Note that, if $P' \geq 0$ this property holds.}
\footnote{Note that this condition is equivalent to $P'(L) \leq \frac{P(L)}{L}$, $\forall L$.}
3. \( \lim_{t \to +\infty} \frac{P(L_t)}{L_t} - 1 = 0 \)

The growth rate of population tends to zero as time tends to infinity.

4. There exists \( L_\infty \) such that \( L_t \leq L_\infty \) for all \( t \) and \( \lim_{t \to +\infty} L_t = L_\infty \)

That is, the population is bounded and converges to \( L_\infty \).

Some well known examples of population laws that verify these properties (for a range in the value of the parameters; see [10]) are described in Table 1:

Table 1: Example of population laws in discrete time

<table>
<thead>
<tr>
<th>Population law</th>
<th>( P(L_t) )</th>
<th>( L_\infty )</th>
<th>( P'(L_\infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beverton-Hold [8]</td>
<td>( \frac{aL_t}{1 + bL_t} )</td>
<td>( \frac{a-1}{b} )</td>
<td>( \frac{1}{a} )</td>
</tr>
<tr>
<td>Ricker [46]</td>
<td>( aL_t e^{-bL_t} )</td>
<td>( \frac{\log(a)}{b} )</td>
<td>( 1 - \log(a) )</td>
</tr>
<tr>
<td>Hassell [36]</td>
<td>( \frac{aL_t}{(1+bL_t)^c} )</td>
<td>( \frac{a_c-1}{b} )</td>
<td>( 1 - c(\frac{a_c-1}{a_c}) )</td>
</tr>
<tr>
<td>Verhulst [51]</td>
<td>( L_t e^{(1-\frac{K}{L_t})} )</td>
<td>( K )</td>
<td>( 1 - r )</td>
</tr>
<tr>
<td>Richards [45]</td>
<td>( L_t + rL_t \left[ 1 - \left( \frac{L_t}{K} \right)^c \right] )</td>
<td>( \hat{K} )</td>
<td>( 1 - rv )</td>
</tr>
<tr>
<td>Logistic</td>
<td>( rL_t \left[ 1 - \left( \frac{K}{L_t} \right) \right] )</td>
<td>( \frac{K(r-1)}{r} )</td>
<td>( r(1 - \hat{K}) )</td>
</tr>
</tbody>
</table>

This paper analyzes the model of Mankiw-Romer-Weil when the hypothesis of exponential growth is modified by introducing a law of population that satisfies the properties listed above. This assumption is congruent with the theories of stable populations (one of the most accepted demographic theories, developed by Lotka [40]). From the empirical point of view constitutes a hypothesis more adjusted to the stylized facts and is usually used by demographers to make projections (for example, Ordirica[43] uses the logistic function to project the size of the world population in 2050).

In developing a dynamic model, a fundamental decision is how to model time. You have to choose between a model in continuous time or discreet time. In economic theory -and particularly in macroeconomics- it is not usual to find a justification for such an election. Generally the choice is arbitrary and depends on technical convenience. It is commonly believed that the qualitative behavior of a dynamic model does not depend on how time is modeled. (see [37]) One can find models that are quantitatively or qualitatively different depending on whether variables are exposed in continuous or discrete time (see [47]). As shown in [24], the choice between the two types of model is not arbitrary and the results and policy recommendations may vary from one version to another. The studies [5], [9], [39], [22], and [23] present some cases where the choice between continuous and discrete time affects the dynamics of the analyzed economic models. In particular, these papers analyze how the stability properties of equilibrium can change depending on the time representation.
The reformulation of classical growth models, considering alternative to exponential population laws, has already been studied for the Solow model (using a Richards equation \cite{7}, using von Bertalanffy law \cite{12}, using logistic law \cite{17}, bounded population \cite{15}, general population laws that verifies the properties mentioned above \cite{13}) and the model of Ramsey (using Richards law \cite{2}, \cite{20} von Bertalanffy law \cite{3}, \cite{29}, \cite{30}, \cite{31}, \cite{32}, or using logistic law \cite{4}, \cite{19}, \cite{25}, \cite{28}, \cite{33}, \cite{34}, \cite{1}, or general population law \cite{26}, \cite{27},\cite{11} ). More recently, the paper by \cite{49} studies the dynamics of a growth model with two types of agents, with different propensities for saving and the work force is modeled by the logistic equation. And the paper \cite{21} where it compares the dynamic behavior in the framework of the Ramsey model if the population grows exponentially or using the logistic model. This paper generalizes \cite{35}, in discrete time, where the author modifies the model by introducing the logistic law of population. This model was developed by Mankiw, Romer, and Weil in 1992 is one of the most influential and widely cited papers in the abundant empirical growth literature. Its publication signified the resurgence of neoclassical growth models in the 1990s. When considering a broader capital definition, the model predicts a lower rate of convergence\footnote{or $\beta$–convergence, defined as the time it takes the economy to reach equilibrium economy} to equilibrium than the Solow model. This implies that the speed of convergence is lower, and that the Mankiw-Romer-Weil model adjust better to empirical data than the original Solow model. This result, coupled with the emergence of endogenous growth models, promoted the development of an empirical line of research focused on convergence and the dispersal ($\sigma$–convergence) of per capita product between countries, group of countries or regions within the same country. In all these works the Mankiw-Romer-Weil model is a fundamental pillar. Given that the introduction of an alternative law of population growth implies changes in the speed of convergence to the equilibrium, the aim is that the present study can be give a contribution to this empirical line of research.

The paper is organized as follows. In section 2, the model is presented. The existence and uniqueness of nonzero equilibrium are showed and the stability of the equilibrium is analyzed. In section 3 the modified model is analyzed. Section 4 studies the speed of convergence and transitional dynamics of the model. Finally, section 5 presents some concluding remarks.

## 2 The model

### 2.1 The original Mankiw-Romer-Weil model

Let begin by introducing the original Mankiw-Romer-Weil in discrete time model with exponential population growth law and analysing the main dynamical properties of the model. (see \cite{41} for a detailed description)

Let consider a closed economy, with a single productive sector, which uses physical
capital ($K_t$), labor force ($L_t$) and human capital ($H_t$) as factors of production ($Y_t$). The economy is endowed with a technology defined by a Cobb-Douglas production function with constant returns to scale:

$$Y_t = K_t^\alpha H_t^\beta L_t^{1-\alpha-\beta}, \quad \alpha, \beta, \alpha + \beta \in (0, 1)$$

The capital stock changes equal the gross investment $I_{Kt} = s_k Y_t$ minus the capital depreciation $\delta K_t$:

$$K_{t+1} - K_t = s_k Y_t - \delta K_t \quad (2)$$

The human capital stock changes equal the gross investment $I_{Ht} = s_h Y_t$ minus the capital depreciation $\delta H_t$:

$$H_{t+1} - H_t = s_h Y_t - \delta H_t \quad (3)$$

The model assumes that the population grows at a constant rate $n > 0$:

$$\begin{cases} L_{t+1} = (1 + n)L_t \\ L(0) > 0 \end{cases} \quad (4)$$

The production function can be expressed in per capita as:

$$\frac{Y_t}{L_t} = \frac{K_t^\alpha H_t^\beta L_t^{1-\alpha-\beta}}{L_t} = \left( \frac{K_t}{L_t} \right)^\alpha \left( \frac{H_t}{L_t} \right)^\beta = y_t \quad (5)$$

If we define $K/L = k$ as the physical capital per worker and $H/L = h$ as the human capital per worker. Then the production per capita is:

$$y_t = k_t^\alpha h_t^\beta \quad (6)$$

Note that

$$\frac{K_{t+1} - K_t}{L_t} = \frac{s_k K_t^\alpha H_t^\beta L_t^{1-\alpha-\beta}}{L_t} - \delta K_t \quad (7)$$

$$\frac{K_{t+1}}{L_{t+1}} \frac{L_{t+1}}{L_t} - \frac{K_t}{L_t} = s_k k_t^\alpha h_t^\beta - \delta k_t \quad (8)$$

$$k_{t+1}(1 + n) - k_t = s_k k_t^\alpha h_t^\beta - \delta k_t \quad (9)$$

The equation of motion for the model which describes how physical capital per worker varies over time:

$$k_{t+1} = \frac{s_k k_t^\alpha h_t^\beta + (1 - \delta)k_t}{1 + n} \quad (10)$$
By a similar reasoning, we arrive to the equation of motion for the model which describes how human capital per worker varies over time:

\[
h_{t+1} = s_h k_t^\alpha h_t^\beta + (1 - \delta) h_t \frac{1}{1+n}
\]

(11)

Then the two dimensional dynamical system:

\[
\begin{cases}
k_{t+1} = s_k k_t^\alpha h_t^\beta + (1 - \delta) k_t \\
h_{t+1} = s_h k_t^\alpha h_t^\beta + (1 - \delta) h_t 
\end{cases} + \frac{1}{1+n}
\]

(12)

describes the dynamics of the model.

Note that the non trivial equilibrium is the point \((k^*, h^*)\) such that:

\[
\begin{cases}
k^* = \left[\frac{s_k^{1-\beta} s_h^\beta}{\delta + n}\right]^{\frac{1}{1-\alpha-\beta}} \\
h^* = \left[\frac{s_h^{1-\alpha} s_h^\alpha}{\delta + n}\right]^{\frac{1}{1-\alpha-\beta}}
\end{cases}
\]

(13)

and the equilibrium of the product is:

\[
y^* = (k^*)^\alpha (h^*)^\beta = \left[\frac{s_k}{\delta + n}\right]^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{s_h}{\delta + n}\right]^{\frac{\beta}{1-\alpha-\beta}}
\]

Then the equilibrium values of long-term capital (physical and human) and product, depend positively on the savings rates \((s_k, s_h)\) and on the degree of efficiency of scale of reproducible factors \((\alpha, \beta)\) and negatively on the rate of depreciation \((\delta)\) and on population growth \((n)\).

In order to analyze the stability of the stationary state we consider the linear approximation of the function \(G : \mathbb{R}^2 \to \mathbb{R}^2\) given by

\[
G(k, h) = \left(\frac{s_k k^\alpha h^\beta + (1 - \delta) k}{1+n}, \frac{s_h k^\alpha h^\beta + (1 - \delta) h}{1+n}\right)
\]

Which gives the first order approximation of the model as:

\[
\begin{pmatrix}
k_{t+1} \\
h_{t+1}
\end{pmatrix} = G(k, h)
\]

The transitional dynamic around the equilibrium \((k^*, h^*)\) can be quantified from the linearization of the system:

\[
\begin{pmatrix}
k_{t+1} \\
h_{t+1}
\end{pmatrix} = G(k^*, h^*) + J_G \begin{pmatrix} k_t - k^* \\
h_t - h^*
\end{pmatrix} = \begin{pmatrix} k^* \\
h^*
\end{pmatrix} + J_G \begin{pmatrix} k_t - k^* \\
h_t - h^*
\end{pmatrix}
\]

where \(J_G\) is the Jacobian matrix of \(G\) evaluated in the equilibrium.
The characteristic polynomial of the Jacobian matrix is:

\[ Q(X) = \left(1 - \delta\right) \left[(n + \delta)(\alpha + \beta) + 1 - \delta\right] \left(1 + n\right)^2 - \left(\frac{(n + \delta)(\alpha + \beta) + 2(1 - \delta)}{1 + n}\right) X + X^2 \]

that presents two positive less than one eigenvalues: \( \lambda_1 = \frac{1 - \delta}{1 + n} \) \( \lambda_2 = \frac{(n + \delta)(\alpha + \beta) + 1 - \delta}{1 + n} \). This implies that the equilibrium is a global attractor.

**Remark 1.** The convergence rate is determined by the higher eigenvalue in absolute value, i.e., \( \lambda_2 = \frac{(n + \delta)(\alpha + \beta) + 1 - \delta}{1 + n} \). One of the characteristics of the model is that the convergence rate is lower than in the Solow model\(^4\). This implies that the M-R-W model fits better to empirical data that the Solow model (see [6] chap. 1).

### 2.2 The modified Mankiw-Romer-Weil discrete time model with a decreasing population growth rate

In the previous model, we replace the growth population law \( L_{t+1} = (1 + n)L_t \) by a law \( L_t = P(L_t) \) which verifies the aforementioned properties, and redoing the step of the previous subsection, the dynamical system describing the modified model can be represented by the following system of differential equation of order 3:

\[
\begin{cases}
k_{t+1} = \frac{s_k k_t^\alpha h_t^\delta + (1-\delta)k_t}{P(L_t)/L_t} \\
h_{t+1} = \frac{s_h k_t^\alpha h_t^\delta + (1-\delta)h_t}{P(L_t)/L_t} \\
L_{t+1} = P(L_t)
\end{cases}
\]  

(14)

Note that the modified model is represented by a system of difference equations of order three.

### 3 Equilibria and stability: qualitative analysis

#### 3.1 The positive steady state

This section investigates the dynamic behaviour of the model’s solution \( (k_t, h_t, L_t) \).

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\(^4\)The speed of convergence that Solow model predicts is: \( \frac{\alpha(n+\delta)+1-\delta}{1+n} \), see Appendix.
Lemma 2. If excluded the trivial solutions obtained by considering $k = 0$, $h = 0$ and $L = 0$, then the model has a unique positive equilibrium $(k^*, h^*, L^*)$ verifying:

$$
\begin{align*}
    k^* &= \left[ \frac{s^1_k - \beta s^1_h}{\delta} \right] \frac{1}{1 - \alpha - \beta} \\
    h^* &= \left[ \frac{s^1_{1-k} - \alpha s^1_h}{\delta} \right] \frac{1}{1 - \alpha - \beta} \\
    L^* &= L_\infty
\end{align*}
$$

Proof. The proof is immediate from solving the system (14) looking for a constant solution. \qed

Remark 3. The values for $k^*$, $h^*$ and $y^*$ match with the ones of the original model of Mankiw-Romer-Weil when $n = 0$. This implies that the equilibrium values of the modified model are higher than those of the original model. And that then the parameters of the population law do not enter in the determinants of $k^*$, $h^*$ and $y^*$. The steady state values of physical capital and human capital depend only on the parameters of technology $\alpha$, $\beta$ and $\delta$ and the exogenous savings rates $s_k$ and $s_h$. This is an important difference with the original model, where an increase in the intrinsic rate of population growth leads to lower levels of these variables in the long run. In addition, given that in the modified model the population size is bounded by the carrying capacity $L^* = L_\infty$, then the aggregate quantity of physical and human capital in the long run are finite and equals $K^* = L_\infty k^*$ and $H^* = L_\infty h^*$ respectively (while in the original Mankiw-Romer-Weil model are infinite).

Proposition 4. The equilibrium point $(k^*, h^*, L^*)$ is a global attractor.

Proof. To analyze the stability of the steady state solution, let consider the linear approximation of the function $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$
T(k, h, L) = \left( \frac{s_k k^\alpha h^\beta + (1 - \delta)k}{P(L)/L}, \frac{s_h k^\alpha h^\beta + (1 - \delta)h}{P(L)/L}, P(L) \right)
$$

around the equilibrium point $(k^*, h^*, L^*)$. The Jacobian matrix of the linear approximation is given by:

$$
\mathbb{J}_T = \begin{pmatrix}
    \delta (\alpha - 1) + 1 & k^* \left( \frac{1 - P'(L_\infty)}{L_\infty} \right) \\
    \frac{\beta s_k \delta}{s_h} & \delta (\beta - 1) + 1 & \frac{1 - P'(L_\infty)}{L_\infty} \\
    0 & \frac{\alpha s_h \delta}{s_k} & P'(L_\infty)
\end{pmatrix}
$$

Then the characteristic polynomial of this matrix is given by

$$
R(X) = (P'(L_\infty) - X) \left( (\delta (\alpha - 1) + 1 - X) (\delta (\beta - 1) + 1 - X) - \alpha \beta \delta^2 \right)
$$
Where the three eigenvalues of the matrix \( J \) takes the form
\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
\]

is a global attractor.

4 Translational dynamics and speed of convergence

The Mankiw-Romer-Weil model is a good approximation to real data, empirically proved to be more robust, better suited to the empirical data, than the Solow model, but it describes the economic reality incompletely. The modified model has richer dynamics, have positive growth at the equilibrium levels of physical and human capital.

In this section we provide a quantitative assessment of the speed of convergence of transitional dynamics. The speed depends on the parameters of technology and preferences and can be computed from the matrix \( J_T \) \((k^*, h^*, L_\infty)\). The transitional dynamics around the steady state \((k^*, h^*, L_\infty)\) can be quantified by using the linearization of system (14):
\[
\begin{pmatrix}
k_{t+1} \\
h_{t+1} \\
L_{t+1}
\end{pmatrix}
= \begin{pmatrix}
k^* & \delta(\alpha - 1) + 1 & k^* \\
h^* & \frac{\beta_s \delta}{s_k} & h^* \\
L_\infty & \frac{\alpha_s \delta}{s_k} & L_\infty
\end{pmatrix}
\begin{pmatrix}
k_t \\
h_t \\
L_t
\end{pmatrix}
\]

Where the three eigenvalues of the matrix \( J_T(k^*, h^*, L_\infty) \) are real, positive and given by: \( \lambda_1 = \delta(\alpha + \beta - 1) + 1 < 1, \lambda_2 = 1 - \delta < 1 \) and \( \lambda_3 = P'(L_\infty) < 1 \).

The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) coincide with those in the standard model when the rate of population growth is zero. The eigenvalue \( \lambda_3 = P'(L_\infty) \) corresponds to the speed of convergence of population to the carrying capacity \( L_\infty \). Each eigenvalue corresponds to one source of convergence and each stable transition path to the steady state of the system takes the form
\[
\begin{align*}
k_t &= k^* + C_1 v_{11} L_t^1 + C_2 v_{21} L_t^2 + C_3 v_{31} (P'(L_\infty))^t \\
h(t) &= h^* + C_1 v_{12} L_t^1 + C_2 v_{22} L_t^2 + C_3 v_{32} (P'(L_\infty))^t \\
L(t) &= L_\infty + (L_0 - L_\infty) (P'(L_\infty))^t
\end{align*}
\]

where \( C_1, C_2, C_3, v_{11}, v_{21}, v_{31}, v_{12}, v_{22} \) and \( v_{32} \) depends on the initial conditions and coefficients of \( J_T(k^*, h^*, L_\infty) \). Then the speed of convergence of physical and human capital depends on eigenvalues \( \lambda_1 = \delta(\alpha + \beta - 1) + 1 \) and \( P'(L_\infty) \). Note that, being population given exogenously, the speed of convergence of population only depends on \( P'(L_\infty) \). In fact, the transition depends on eigenvalue with higher absolute value. If \( |P'(L_\infty)| < |\lambda_1| \), then the speed of convergence of \( L_t \) is faster than that of \( k_t \) and \( h_t \) and if \( |P'(L_\infty)| > |\lambda_1| \) then all variables converge at speed \( P'(L_\infty) \).
Remark 5. Regardless of whether the speed of convergence is $\lambda_1$ (it just depends on the degree of efficiency of scale reproducible factor and on the rate of depreciation) or $\lambda_3$ (it just depends on the population law), in both cases it is lower than in the original model.

5 Concluding Remarks

In economic growth theory it is usually assumed that population growth follows an exponential law. This is clearly unrealistic because it implies that population goes to infinity when time goes to infinity. In this study an improved version of the Mankiw-Romer-Weil growth model is developed by introducing a general population law.

The model is presented as a dynamical system of dimension three, who supports a unique non-trivial equilibrium, which is a global attractor. In the modified model the equilibrium values of the product, physical capital and human capital per capita depend on the degree of efficiency of scale of reproducible factors ($\alpha, \beta$), depreciation rate ($\delta$) and the savings rate ($s_k, s_h$), but do not depend on the parameters of the population. In addition, their values are higher than the classical model, for any value of the constant rate $n > 0$ of population growth.

In the equilibrium of the classical Mankiw-Romer-Weil model, aggregate physical and human capital tends unrealistically to infinity as $t$ tends to infinity, because population grows to infinity. This situation is improved in the modified model, where in equilibrium aggregate physical and human capital tends to the more realistic finite values $K^* = L_\infty k^*$ and $H^* = L_\infty h^*$.

Finally, the study shows that the model has a finite speed of convergence that depends on technology parameters and the rate of depreciation or the law of population, but not on both. In addition, this velocity is always smaller than in the original model.

Future research can include modelling population by an equation that depends on other variables of the model; i.e., to endogenize population. An alternative line of research involves the introduction of a particular law of population to find the closed solutions of the model.

A third line of research can include the possibility of migration.

Finally, an alternative line of investigation is the empirical study under an econometric specification that follows the modified model.

6 Appendix

6.1 The speed of convergence in the Solow model

The product per worker, considering a production function of Cobb-Douglas, is given by:
\[ y_t = k_t^\alpha \]  

(19)

The equation of motion for the model which describes how physical capital per worker varies over time:

\[ k_{t+1} = \frac{sk_t^\alpha + (1 - \delta)k_t}{1 + n} \]  

(20)

In this framework the equilibrium of the product is:

\[ k^* = \left( \frac{s}{n + \delta} \right)^{\frac{1}{1 - \alpha}} \]  

(21)

Linealizing equation 20 in the vicinity of the equilibrium is obtained:

\[ k_{t+1} = k^* + F'(k^*)k_t \]  

(22)

where

\[ F(k) = \frac{sk^\alpha + (1 - \delta)k}{1 + n} \]  

(23)

and

\[ F'(k^*) = \frac{\alpha(n + \delta) + 1 - \delta}{1 + n} \]  

(24)

given that:

\[ 0 < F'(k^*) < 1 \]  

(25)

This implies that the equilibrium is a global attractor and the speed of convergence that Solow model predicts is: \[ \frac{\alpha(n + \delta) + 1 - \delta}{1 + n} \]

References


