Economic growth and human capital accumulation: a discrete time analysis*

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Abstract

This paper reformulates the classical Razin model of economic growth and human capital accumulation by representing time as a discrete variable. In addition, the model is developed in a more general framework of the Ramsey neoclassical model of optimal economic growth. The study examines the optimal trajectories in human and physical capital and consumption showing the existence of a unique steady state which stability is analyzed. The paper compares the results with the original study of Razin.

Keywords: economicdynamics; human capital; economic growth; discrete time.

JEL classification: C62; O41.

1 Introduction

Modern macroeconomics is founded on the Neoclassical growth model, and owes its origins in the developments of Ramsey (1928), Solow (1956), Swan (1956), Cass (1965), and Koopmans (1965). This model has the virtue of representing in a simple framework the most relevant stylized facts on economic growth, which where pointed by Kaldor (1961) and more recently improved by Jones and Romer (2009). Among its main simplifying assumptions, the Neoclassical growth model builds on the assumption that growth occurs through the accumulation of physical capital. Think only in the investment of physical

*This research work was supported by CSIC-UDELAR (project "Grupo de investigacin en Dinmica Econmica"; ID 881928.

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capital was an important assumption of the first neoclassical model. Probably correct because the central role of accumulation itself fifty or sixty years ago. From this years at today, growth theorists working hard over other endogenous state variables excluded from consideration by the initial neoclassical setup. Ideas, institutions, population, and human capital are now at the center of growth theory (Jones and Rommer, 2009).

A vast literature covering at least the last six decades shows that investment in the human factor may well have a higher payoff in terms of increased output than does any other input, and find evidence supporting that higher levels of human capital contribute to reduce income inequality and improve living standards -Gunder Frank (1960), Lucas (1988), Glomm and Ravikumar (1992), Romer (1990), Mankiw, Romer and Weil (1992), Barro and Sala-i-Martin (1995, Ch. 5), Barro (1997) and Barro (2013). However, not all the empirical literature points in this direction. Other studies that examined the relationship between changes in human capital accumulation -measured through years of schooling- and changes in average incomes failed to find a significant association -Benhabib and Spiegel (1994), Pritchett (2000) and Shekhar (2006).

The formal theoretical developments on this facts can be registered under the term human capital models. The main purpose of these models is to understand which factors affect human capital investments and how these influence the process of economic growth and economic development. Human capital refers to all the attributes of workers that potentially increase their productivity in all or some productive tasks. The human capital theory, primarily developed by Becker (1965), Ben-Porath (1967) and Mincer (1974), is about the role of human capital in the production process and about the incentives to invest in skills, including pre-labor market investments (schooling) and on-the-job investments (training). Seminal works in modeling the accumulation of human capital in the context of growth theory are Uzawa (1965), Razin (1972) and Lucas (1988). Razin (1972) formalize this concepts in the framework of the Ramsey neoclassical model of optimal economic growth.

A first wave of extends for the endogenous growth theory to modelate human capital can be found in the works of Lucas (1988), Tamura (1991), Rebelo (1991), Glomm and Ravikumar (1992), Caballe and Santos (1993), Mulligan and Sala-i-Martin (1993), and Barro and Sala-i-Martin (1995, Ch. 5), where the economy tends toward a steady state ratio of human to physical capital, but the ratio may depart from its long-run value in an initial state. The extent of this departure generally affects the rate at which per capita output approaches its steady-state value. Nelson and Phelps (1966), Romer (1986), Romer (1990), Benhabib and Spiegel (1994), go deep in this topics and investigate the relationship between technological change and the endowment of human capital. All these researchs find their roots in the work of Uzawa (1965), Razin (1972) and Lucas (1988), and also on the work of Arrow (1962) and Sheshinski (1967), those who developed the concept of learning by doing.

In this context, the relevance of the endowment of human capital and its relation with phisical ones becomes more evident. The cited works before underscore an interaction effect through which growth rates are more sensitive to its starting level of per capita output if its initial level of human capital is grater. In the main, this approach provides a theory of technical progress, one of the central missing elements of the neoclassical model. Modelize technological change in the neoclassical framework is difficult, because its violate the standard assumptions of perfect competition. Seminal works who deal with the incorporation of Research and Development (R&D) theories and imperfect competition into the growth framework began with Romer (1987, 1990), Aghion and Hewitt (1992), Grossman and Helpman (1991) and Barro and Sala-i-Martin (1995). In these frameworks, the long-term growth rate depends on governmental actions, whose great potential influence the long-term rate of growth. Technological advance involves the creation of new ideas, which are partially nonrival and therefore have aspects of public goods. So, the returns to scale tend to be increasing if the nonrival ideas are included as factors of production, and perfect competition is no more valid (Barro, 1997).

In sum and following Barro (2013), the recent endogenous-growth models are useful for understanding why advanced economies (and the world as a whole) can continue to grow in the long run despite the workings of diminishing returns in the accumulation of physical and human capital. In contrast, the extended neoclassical framework does well as a vehicle for understanding relative growth rates across countries.

A particular fact is that in all the cases cited above, the neoclassical growth model is developed in continuous time (whith the exceptions of Glomm and Ravikumar (1992) and Tamura (1991), who work on a overlapping generations model). As pointed by Turnowsky (1977), in the modeling of dynamic economic systems, the choice of time representation is one fundamental methodological matters that arises when one aims to formulate the model. Discrete time and continuous time are two alternative frameworks to model variables that evolve over time. Some economists argue that discrete-time modeling should be used because decisions are made and data are released at discrete intervals. Others, however, argue that life unfolds continuously suggesting that a continuous time framework is more realistic (see Krivine et als. 2007; Giannitsarou and Anagnostopoulos, 2005). In addition, most of the economists choose continuous time formulation mainly because they find it to be more tractable and often more transparent. But to some degree this choice is a question of taste. (See Bosi and Ragot (2009)). From the mathematical viewpoint, time is a discrete variable when is represented by a numerable set of values; i.e., when there exists a bijective correspondence with the set of natural numbers. On the other hand, time is a continuous variable when can be represented by an interval of the set of real numbers; ie.e, when there exists a bijection with the real line. As pointed by Romer (2012; page 12) when presenting the Solow model, "the choice between continuous and discrete time is usually based on convenience. For example, the Solow model has essentially the same implications in discrete as in continuous time, but is easier to analyze in continuous time."

Discrete time views variables as occurring at separate points in time; i.e., time is viewed as a discrete variable. In discrete time models, each variable of interest is measured once at each time period that is determined by the tick of a clock. The number of measurements between any two time periods is finite. Discrete-time signals are typically made at sequential integer values of the variable time. Continuous-time signals are determined along a continuum of time and are thus represented by a continuous independent variable. Whenever one writes down a model some fundamental choices have to be made. One of these is the question of discrete versus continuous time modeling. Discrete time models make use of difference equations, and continuous time make use of differential equations. Discrete time is often employed when empirical measurements are involved. On the other hand, sometimes it is more mathematically tractable to construct theoretical models in continuous time. In fact, most of theoretical models, especially in the growth literature, are built in continuous time. Preference for system of differential equations comes essentially from technical considerations. As pointed by Bosi and Ragot (2009), in the well-known book in Economic Dynamics (Gandolfo, 1997) the author puts forward eight arguments in favor of a continuous instead of a discrete-time representation of economic activity.

In economic theory -and particularly in macroeconomics- it is commonly believed that the qualitative behavior of a dynamic model does not depend on whether time is formulated as a continuous variable or a discrete variable. In general this is not true. Time modeling is neither trivial nor neutral and has economic consequences. The choice between continuous-time and discrete-time formulations is usually arbitrary and depends mostly on technical convenience. (See Jarrow and Protter (2012); Miao (2014)) One can find models that are quantitatively or qualitatively different depending on whether variables are exposed in continuous or discrete time. (See Stutzer (1980); Bosi and Ragot (2009)) As discussed in Gonzalez and Pecha, (2010), the choice between the two types of model is not arbitrary and can determine the results independently of the underlying economic mechanisms. In particular, economic conclusions and policy recommendations can vary in one version to another. Some recent papers, including Bambi and Gori (2014), Bosi and Ragot (2012), and Gomez (2014) discuss how the choice between continuous and discrete time affects the dynamics of economic models. In particular, these papers analyze how the stability properties of equilibria can change depending on the time representation. Other papers, including Licandro and Puch (2006) and Medio (1991) discuss the temporal dimension in economic models.

The main purpose of this paper is to extend and reformulate the classical Razin (1972) model of optimal economic growth and investment in human capital in two ways. First, examine a discrete-time model and demonstrate that this timing assumption has an important implication on equilibrium determinacy. Second, consider a more general framework by expressing the model in the context of the Ramsey neoclassical model of optimal economic growth (Ramsey (1928); Cass (1965) and Koopmans (1965)).

The paper is organized as follows. In section 2, the model is presented. The separation theorem that allows to unfold the optimization problem into two separate exercises is analyzed in section 3. Section 4 studies the existence and uniqueness of nonzero equilibrium and the stability of the steady state in human capital. The next section examines the optimal trajectories of capital and consumption. Section 6 studies the equilibria an satibility of the entire model. Finally, section 7 presents some concluding remarks.

2 The Model

The general framework of Ramsey model considers an economy that produces a unique good, with convex preferences and technology and with no external effects of any kind, that can be consumed or used (along with homogeneous labour) as capital in the production. This model is one of the basic workhorse models in macroeconomics. The original version of this model is a continuos-time model and is described by a two dimensional system of differential equations, whose unique non-trivial steady-state equilibrium is a stable saddle-point. Along this stable balanced path, the rate of growth of per-capita consumption and per-capita capital growth at a common constant rate, and this is simply proportional to the given rate of technical change. This optimum is also the unique competitive equilibrium program, provided that consumers and firms have rational expectations about future prices (Lucas, 1988).

In terms of the model fit to the data, for growing economies, it is necessary to make some comments. The growth of labor productivity and capital per-capita, the stability of the real interest rate or return on capital and the ratio of capital to output, and the stable shares of capital and labor respect to national income, are correctly predicted by the model (Kaldor, 1961 and Jones and Rommer, 2009).

However, secular changes in manhours per household that the model assumes away, and labor's share is secularly rising, not constant as assumed, are some of the quantitative results that drive to extend or adjust the neoclassical growth model. Beyong this, qualitatively and even granted its limitations, this simple neoclassical model has an important meening related to the behavour of the economies and about policy recommendations. In this sense, the model make a distinction between changes in parameters that alter growth rates along balanced paths -growth effects- and changes that raise or ower balanced growth paths without affecting their slope -level effects. Intending to incorporate into the model the determination of the main long-term relationships, at the end of the 1980s endogenous growth models are developed.Endogeneous technological progress and human capital were two of the key features in these models (Barro, 2013).

Following the basic framework presented by Barro and Sala-i-Martin (1995, p. 172 et seq), a model of endogenous growth with human capital can be represented by the following dynamic optimization problem:

$$Max \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$\begin{aligned} s.t.: \qquad & N_{t+1} = (1+n)N_t \\ & A_{t+1} = (1+\alpha_t)A_t \\ & L_t = (1-g(\alpha_t))N_t \\ & I_t = K_{t+1} - K_t + \delta K_t \\ & H_t = A_{t+1}L_{t+1} - A_t L_t + \gamma A_t L_t \\ & C_t + I_t + H_t \leq F(K_t, A_t L_t) \end{aligned}$$

Where capital letters C, N, L, K, A, I, H represent the aggregate consume, population, work force, phisical capital, human capital, investment in phisical capital and investment in human capital respectively. Equations for human capital, work force and investment in human capital are added respecto to the basic Ramsey model. This formulation can be named as the Uzawa-Razin-Lucas (URL) model. U is the overall utility function, F is the production tecnology function that equalice the flow of output Y. Firms are identical and have the same constant return to scale, and F are represented by the typical neoclasic formulation: C^1 class, strictly increasing, homogeneous of degree one and strictly cuasi-concave, with marginal products F_K, F_A, F_N positive, decreasing and satisfying the Inada conditions:

$$\begin{aligned} &\lim_{K\to 0} F_K = \lim_{A\to 0} F_A = \lim_{N\to 0} F_N = \infty \\ &\lim_{K\to\infty} F_K = \lim_{A\to\infty} F_A = \lim_{N\to\infty} F_N = 0 \end{aligned}$$

For instance, g represent the fraction of time devoted to education ($0 \leq g \leq 1$), who depend of α_t, the relative rate of change over time of the amount of the skill possessed by labor (human capital). It is assumed, just like Razin (1972), diminishing returns to education, so the function $g(\alpha_t)$ is increasing and strictly convex. Population growth at a constant rate n and β is the discunt factor such taht $0 < \beta < 1$.

The URL extension of the Ramsey model is based on several assumptions. Households in the economy, with one or more adult who provide labor services in exchange of wages (constant), receive (constant) interest rate on assets, purchase goods for consumption and save by accumulating new assets. All households are identical and their preferences are the preferences attributed to the social planner. The function $U : \Re_+ \to \Re$, has the usual properties of twice continuosly differentiable with $U(0) = 0$, strictly increasing on the consumption ($U' > 0$) and strictly concave ($U'' < 0$).

3 Separability of the optimization problems

Presented the general framework, the goal is to determine the existence and caracteristics of the optimun paths for the three main variables C, α (and therefore A) and K. To do this, and similarly to was done by Razin, it follow the approach of Acemoglu (2009) -but in discrete time-, namely *a simple separation theorem.* Acemoglu (2009) demostrate that, whith perfect capital markets, schooling decisions maximize the net present discounted value of the individual and can thus be separated from consumption decisions. The intuition behind this “two step” optimization strategy is explined below.

Assume that the household does not enjoy either the time spent on education or the possession of a stock of human capital directly. Its utility is derived from the enlarged stream of consumption made possible by investing in education. Therefore the problem of specifying the paths of consumption and the rate of investment in human capital that yield the highest attainable (intertemporal) utility can be separated into two sub-problems. The first is to maximize human wealth by an appropriate policy of investment in human capital. Given this amount of human wealth, the second problem is to achieve the highest level of inter temporal welfare by an optimum consumption and saving policy.

4 Human capital steady rate

In this section it will be shown that the optimum path for the investment in human capital is to invest at a steady rate (named α).

The problem is as follows: given the investment in human capital at time t, A_t, find A_{t+1} such as

$$Max \sum_{t=0}^{\infty} \rho^t w N_t A_t \left(1 - g(\alpha_t)\right)$$

is maximum, restricted to:

$$\textit{subject to:} \qquad \begin{array}{l} \alpha_t A_t = A_{t+1} - A_t \\ n N_t = N_{t+1} - N_t \end{array} \tag{1}$$

Where all the asumptions presented before are remaining and it is added w as the hour wage for the labor services. In addition, let $0 < \rho = \frac{1}{1+r} < 1$ where r is the interest market rate, and define the maximum and minimum rates of investment in human capital by $g(\overline{\alpha}) = 1$ and $g(\underline{\alpha}) = 0$, respectively. So the relative rate of change of the amount of skills varies over $\left[\overline{\alpha}, \underline{\alpha}\right]$, were it is asumed $\underline{\alpha} \geq 0$, because otherwise ($\alpha$ negative) imply disaccumulation (or destruction) of stock of human capital. Finally, for analytcal convenience, asume $g'(\underline{\alpha}) = 0$.

Let $V(A_0)$ the maximum value of the objective function:

$$V(A_0) = Max \sum_{t=0}^{\infty} \rho^t w N_t A_t \left(1 - g(\alpha_t)\right)$$

First notice that the difference equation for N_t can be solved:

$$N_t = N_0(1+n)^t$$

Substituting this result in our maximization problem it get:

$$V(A_0) = Max \sum_{t=0}^{\infty} \rho^t (1+n)^t w N_0 A_t \left(1 - g(\alpha_t)\right)$$

$$s.t.: \qquad \alpha_t A_t = A_{t+1} - A_t$$

where A_t is the state variable, and A_{t+1} and α_t are the control variables.
As the control variables are related via the restriction, it can choose either one to work with. Lets choose A_{t+1}. Then:

$$V(A_0) = \max_{\{\alpha_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \rho^t (1+n)^t w N_0 A_t \left(1 - g(\alpha_t)\right)$$

$$s.t. : \qquad \alpha_t A_t = A_{t+1} - A_t$$

Define $\tilde{\rho}^t = \rho^t (1+n)^t = \left(\frac{1+n}{1+r}\right)^t$ and $u(A_t, \alpha_t) = w N_0 A_t \left(1 - g(\alpha_t)\right)$, the Bellman equation at time s associated with this problem is:

$$V(A_s) = \max_{\{\alpha_s, A_{s+1}\}} \left\{ u(A_s, \alpha_s) + \tilde{\rho} V(A_{s+1}) \right\}$$

$$s.t: \qquad \alpha_s A_s = A_{s+1} - A_s$$

In terms of the variables A_s and A_{s+1}, it is possible to rewrite this equation as follows:

$$V(A_s) = \max_{\{A_{s+1}\}} \left\{ u(A_s, \frac{A_{s+1}}{A_s} - 1) + \tilde{\rho} V(A_{s+1}) \right\}$$

The First Order Condition (FOC) for this equation is to derivate the right-hand expression of the equation respect to A_{s+1} and equalize to zero:

$$\frac{\partial V(A_s)}{\partial A_{s+1}} = \frac{\partial u(A_s, \frac{A_{s+1}}{A_s} - 1)}{\partial A_{s+1}} + \tilde{\rho} V'(A_{s+1}) = 0$$

Then, it is:

$$\frac{\partial u(A_s, \frac{A_{s+1}}{A_s} - 1)}{\partial A_{s+1}} = -\frac{w N_0 A_s g'(\alpha_s)}{A_s} = -w N_0 g'(\alpha_s)$$

Hence, the FOC can be written as follow:

$$\boxed{V'(A_s) = u'(A_s, \alpha_s)\frac{1}{A_s} + \tilde{\rho} V'(A_{s+1}) = 0}$$

Or, in equivalent form:

$$\boxed{V'(A_s) = -w N_0 g'(\alpha_s) + \tilde{\rho} V'(A_{s+1}) = 0}$$

Once A_{s+1} that satisfies the Bellman equation is found, there exists a policy function $A_{s+1} = d(A_s)$ which in turn satisfies:

$$V(A_s) = u(\alpha(A_s, d(A_s)), A_s) + \tilde{\rho} V(d(A_s))$$

where $\alpha(A_s, d(A_s)) = \frac{d(A_s)}{A_s} - 1$, so:

$$V(A_s) = u(\frac{d(A_s)}{A_s} - 1, A_s) + \tilde{\rho} V(d(A_s) = w N_0 A_s \left(1 - g(\frac{d(A_s)}{A_s} - 1)\right) + \tilde{\rho} V(d(A_s)$$

Derivating the last equation respect to the state variable A_s is obtained:

$$V'(A_s) = w N_0 \left\{ 1 - g(\frac{d(A_s)}{A_s} - 1) - A_s g'(\frac{d(A_s)}{A_s} - 1) \left[\frac{d'(A_s) A_s - d(A_s)}{A_s^2} \right] \right\} + \tilde{\rho} \frac{\partial V(d(A_s))}{\partial d(A_s)} d'(A_s)$$

After some reordering and operations:

$$V'(A_s) = wN_0\left[1-g(\frac{d(A_s)}{A_s}-1)+g'(\frac{d(A_s)}{A_s}-1)\frac{d(A_s)}{A_s}\right]+\left[-wN_0 g'(\frac{d(A_s)}{A_s}-1)+\tilde{\rho}V'(d(A_s))\right]d'(A_s)$$

Notice that the second term between square brackets is the FOC and so equals 0. Hence:

$$V'(A_s) = wN_0\left[1-g(\frac{d(A_s)}{A_s}-1)+g'(\frac{d(A_s)}{A_s}-1)\frac{d(A_s)}{A_s}\right] \tag{2}$$

Advancing one period in (2):

$$V'(A_{s+1}) = wN_0\left[1-g(\frac{d(A_{s+1})}{A_{s+1}}-1)+g'(\frac{d(A_{s+1})}{A_{s+1}}-1)\frac{d(A_{s+1})}{A_{s+1}}\right] \tag{3}$$

Using the optmial policy $A_{s+1} = d(A_s)$ and so $d(A_{s+1}) = A_{s+2}$, is possible to write $\frac{d(A_{s+1})}{A_{s+1}} - 1 = \alpha_{s+1}$ and so rewrite (3) as follows:

$$V'(A_{s+1}) = wN_0\left[1-g(\alpha_{s+1})+g'(\alpha_{s+1})\Big(\alpha_{s+1}+1\Big)\right] \tag{4}$$

Substituting (4) in the FOC, it get:

$$-wN_0 g'(\alpha_s)+\tilde{\rho}wN_0\left[1-g(\alpha_{s+1})+g'(\alpha_{s+1})\Big(\alpha_{s+1}+1\Big)\right]=0$$

So the Euler equation for this problem can be represented by:

$$\boxed{g'(\alpha_s)=\tilde{\rho}\left[1-g(\alpha_{s+1})+g'(\alpha_{s+1})\Big(\alpha_{s+1}+1\Big)\right]} \tag{5}$$

Lemma 1. *The Euler equation (5) admits a constant solution.*

Proof. The objective now is to find a constat solution α that verifies (5). That is, to find $\alpha \in \left[\underline{\alpha},\overline{\alpha}\right]$ that verifies

$$g'(\alpha) = \tilde{\rho}\Big[1 - g(\alpha) + g'(\alpha)\Big(\alpha + 1\Big)\Big] \tag{6}$$

.
Notice that in said case, it would have that $g'(\alpha)\Big(\frac{1}{\tilde{\rho}} - (\alpha+1)\Big) = 1 - g(\alpha)$. As $g'(\alpha)$ and $1 - g(\alpha)$ are non negative, it is must have $\alpha \leq \frac{1}{\tilde{\rho}} - 1$, so it makes sense to consider $\overline{\alpha} \leq \frac{1}{\tilde{\rho}} - 1$.
Let

$$H(\alpha) = \frac{1}{\tilde{\rho}}g'(\alpha) - 1 + g(\alpha) - g'(\alpha)(\alpha + 1)$$

It shuld seek for an α such that $H(\alpha) = 0$. To do so, making the following observation:

- $H(\overline{\alpha}) = g'(\overline{\alpha})\Big(\frac{1}{\tilde{\rho}} - (\overline{\alpha}+1)\Big) \geq 0$, in virtue of the upper bound assumed for $\overline{\alpha}$
- $H(\underline{\alpha}) = g'(\underline{\alpha})\Big(\frac{1}{\tilde{\rho}} - (\underline{\alpha}+1)\Big) - 1 < 0$, for $g'(\underline{\alpha}) = 0$

So, by the intermediate value theorem of Bolzano, it is known that there exists $\alpha \in \Big[\underline{\alpha}, \overline{\alpha}\Big]$ such that $H(\alpha) = 0$. That is, there are a fixed point for the system, and hence the steady solution that it was intended to find. □

The next step in line is to study the stability of said equilibrium. Let start writing the dynamical system (5) in the following form:

$$\Psi(\alpha_s) = \alpha_{s+1} \tag{7}$$

Let define $\Phi(x) = \tilde{\rho}\Big[1 - g(x) + g'(x)(x+1)\Big]$. Notice that $\Phi'(x) = \tilde{\rho}(g''(x)(x+1))$, which is non zero in a neighbourhood of α. Hence it is locally invertible, and defining $\Psi(\alpha_s) = \Phi^{-1}(g'(\alpha_s))$, equation (5) can be expressed in the required form.

Then, to analyze the stability of the equilibrium $|\Psi'(\alpha)|$ must be computed. Note that

$$\Psi'(\alpha) = \Big(\Phi^{-1}g'\Big)'^{-1})'(g'(\alpha)).g''(\alpha) = \frac{g''(\alpha)}{\Phi'(\alpha)} = \frac{1}{\tilde{\rho}(\alpha+1)} \tag{8}$$

If $\overline{\alpha} < \frac{1}{\tilde{\rho}} - 1$, it is $|\Psi'(\alpha)| > 1$ and this imply that the equilibrium is unstable (repulsor).

If $\overline{\alpha} = \frac{1}{\tilde{\rho}} - 1$, it is $H(\overline{\alpha}) = 0$, implying that $\alpha = \overline{\alpha}$, and $\Psi'(\overline{\alpha}) = 1$. Derivating equation (8) and evaluating in $\overline{\alpha}$, we obtain:

$$\Psi''(\overline{\alpha}) = \frac{g'''(\overline{\alpha})}{g''(\overline{\alpha})}.(1 - \tilde{\rho}), \tag{9}$$

which is grater than zero if and only if $g'''(\overline{\alpha}) > 0$. These results imply the following Lemma:

Lemma 2. *There exists an equilibrium α for the system such that $0 < \alpha \leq \frac{1}{\tilde{\rho}} - 1$ and it is unstable. (Unless $\overline{\alpha} = \frac{1}{\tilde{\rho}} - 1$, in which case it is needed that $g'''(\overline{\alpha}) > 0$ for it to be unstable).*

The Euler equation (5) characterize the optimal investment on education and skills plan. It states the relationship between current and future rate of human capital investment, and also depends on the relationship between the rate of population growth and the market interest rate via the parameter $\tilde{\rho}$.

Esto es posible de afirmar?: If the population growth is greater than de market interest rate (i.e. $\tilde{\rho} > 1$), the human capital accumulation decreases and make it on a decreasing rate. In the limit, no investment in human capital take place. In fact, this result is consistent with the condition that $0 < \alpha \leq \frac{1}{\tilde{\rho}} - 1$ for a constant solution. Given that $\tilde{\rho} = \frac{1+n}{1+r}$, a positive α requires that $\frac{1+r}{1+n} - 1 > 0$, and therefore $r > n$.

On the other hand, equation (6) states the condition for the optimal rate of human capital investment. A little algebra shows that the equation can be rewritten as:

$$g'(\alpha) = \frac{\Big(1 - g(\alpha)\Big)\Big(1 + n\Big)}{r - \alpha - n - \alpha n} \tag{10}$$

Equation is essentially the same that equation $(12b)$ in Razin (1972), and both sets the equality of the marginal cost and the marginal benefits of investment in human capital. The case pointed out by Razin where the the marginal cost exceeds the marginal benefit of investment at the edge point (equation 12 a, p. 5 in his paper), increasing costs occurs and no investment takes place, is avoided by the fact that it was assumed g' (α)=0.

5 Capital and Consumption optimal paths

Now return to the original problem:

$$Max \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$s.a.: \qquad \begin{array}{l} N_{t+1} = (1+n)N_t \\ A_{t+1} = (1+\alpha_t)A_t \\ L_t = (1-g(\alpha_t))N_t \\ I_t = K_{t+1} - K_t + \delta K_t \\ H_t = A_{t+1}L_{t+1} - A_tL_t + \gamma A_tL_t \\ C_t + I_t + H_t \leq F(K_t, A_tL_t) \end{array}$$

but now, based on what is established by the separability theorem, considering $\alpha_t = \alpha$, the steady rate found in the previous section.

Due to the convexity of the problem, the solution (if exists) will be on the boundary of the feasible region. That is, to look for a consumption path tath satisfyies:

$$C_t + I_t + H_t = F(K_t, A_tL_t) \tag{11}$$

wich in turn, substituting I_t and H_t from 5 leads to

$$C_t = F(K_t, A_tL_t) - K_{t+1} + (1-\delta)K_t - A_{t+1}L_{t+1} + (1-\gamma)A_tL_t \tag{12}$$

The following is redescribe the model in terms of per capita, but taking into consideration the human capital. That is, introduce the new variables $k_t = \dfrac{K_t}{A_tN_t}$ and $c_t = \dfrac{C_t}{A_tN_t}$.
As it is known, $F(K_t, A_tL_t) = A_tL_t\, f\left(\dfrac{K_t}{A_tL_t}\right)$ and $L_t = (1-g(\alpha_t))N_t$, where it was used $\alpha_t = \alpha$ from the previous section.

With all this taken into consideration, equation (12) turns into:

$$\boxed{\begin{aligned} c_t = (1-g(\alpha)) f\left(\frac{k_t}{(1-g(\alpha))}\right) - k_{t+1}(1+\alpha)(1+n) + k_t(1-\delta) \\ + (1-g(\alpha))[(1-\gamma) - (1+\alpha)(1+n)] \end{aligned}} \tag{13}$$

So, the model in terms of per capita is this:

$$Max \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{14}$$

s.t.:

$$c_t = (1 - g(\alpha^*))f\left(\frac{k_t}{(1 - g(\alpha^*))}\right) - k_{t+1}(1 + \alpha^*)(1 + n) + k_t(1 - \delta^*)$$
$$+ (1 - g(\alpha^*))[(1 - \gamma) - (1 + \alpha^*)(1 + n)]$$

As it considered α given by the previous section, the problem lies only in terms of k_t, k_{t+1}, c_t and c_{t+1}.

The next step is to present the Bellman equation for this problem, which in this case is:

$$V(k_s) = \max_{\{c_s, k_{s+1}\}} \left\{u(c_s) + \beta V(k_{s+1})\right\}$$

The FOC for the problem is:

$$\frac{\partial V(k_s)}{\partial k_{s+1}} = u'(c_s)\frac{\partial c_s}{\partial k_{s+1}} + \beta V'(k_{s+1}) = 0$$

So:

$$\frac{\partial c_s}{\partial k_{s+1}} = -(1 + \alpha)(1 + n)$$

Hence:

$$\boxed{-(1 + \alpha)(1 + n)u'(c_s) + \beta V'(k_{s+1}) = 0} \quad (15)$$

Once found k_{s+1} that satisfies the Bellman equation, there exists a policy function $k_{s+1} = d(k_s)$ which in turn satisfies:

$$V(k_s) = u(c_s) + \beta V(d(k_s)) \quad (16)$$

With

$$c_t = (1 - g(\alpha))f\left(\frac{k_t}{(1 - g(\alpha))}\right) - d(k_t)(1 + \alpha)(1 + n) + k_t(1 - \delta)$$
$$+ (1 - g(\alpha))[(1 - \gamma) - (1 + \alpha)(1 + n)]$$

Now derivate the equation (16) respect to k_t and obtain:

$$V'(k_s) = u'(c_s)\frac{\partial c}{\partial k} + \beta V'(d(k_s))d'(k_s)$$

Where

$$\frac{\partial c_t}{\partial k_t} = f'\left(\frac{k_t}{(1-g(\alpha))}\right) - (1+\alpha)(1+n)d'(k_s) + (1-\delta)$$

So:

$$V'(k_t) = u'(c_t)\,(f'(\frac{k_t}{(1-g(\alpha))})-(1+\alpha)(1+n)d'(k_t) + (1-\delta))) + \beta V'(d(k_t))d'(k_t)$$

Reordering:

$$V'(k_t) = u'(c_t)\left(f'\left(\frac{k_t}{(1-g(\alpha))}\right) + 1 - \delta\right) + \left(-(1+\alpha)(1+n)u'(c_t) + \beta V'(d(k_t))\right)d'(k_t)$$

The second term of the equation is the first order condition, so it equals zero. Hence:

$$V'(k_t) = u'(c_t)\left(f'\left(\frac{k_t}{(1-g(\alpha))}\right) + 1 - \delta\right) \tag{17}$$

Advancing one period in (17):

$$V'(k_{t+1}) = u'(c_{t+1})\left(f'\left(\frac{k_{t+1}}{(1-g(\alpha))}\right) + 1 - \delta\right)$$

Substituting this expression in (15) it is arrived at the Euler equation:

$$\boxed{u'(c_t) = \frac{\beta}{(1+n)(1+\alpha)}u'(c_{t+1})\left(f'\left(\frac{k_t}{(1-g(\alpha))}\right) + 1 - \delta\right)} \tag{18}$$

Equations (18) and (13) define a system of difference equations:

$$\begin{cases} c_t = (1-g(\alpha))f\left(\frac{k_t}{(1-g(\alpha))}\right) - k_{t+1}(1+\alpha)(1+n) + k_t(1-\delta) \\ \qquad + (1-g(\alpha))[(1-\gamma) - (1+\alpha)(1+n)] \\ \\ u'(c_t) = \frac{\beta}{(1+n)(1+\alpha)}u'(c_{t+1})\left(f'\left(\frac{k_t}{(1-g(\alpha))}\right) + 1 - \delta\right) \end{cases}$$

Equation (13) is the percapita budget constrain augmented with human capital. The Euler equation (18) state that the marginal utility of consumption today to be equal to the marginal utility of consumption tomorrow multiplied

by the product of the discount factor and the gross rate of return. This result generalize de common important "consumption Euler" equation for the case of a model with human capital. Notice that in the case of complete especialization in work (i.e. $\alpha = \underline{\alpha}$ so $g(\underline{\alpha}) = 0$) the model reproduce de same equations that the Ramsey model. In this sense, the URL model presented here can be viewed as a generalization of the Ramsey model.

Since u' is invertible, the previous system can be expressed as:

$$\begin{cases} k_{t+1} = \dfrac{(1-g(\alpha))\left[f\left(\dfrac{k_t}{(1-g(\alpha))}\right) + (1-\gamma) - (1+\alpha)(1+n)\right] + k_t(1-\delta) - c_t}{(1+\alpha)(1+n)} \\ \\ c_{t+1} = u'^{-1}\left(\dfrac{u'(c_t)(1+n)(1+\alpha)}{\beta\left(f'(\dfrac{k_t}{(1-g(\alpha))}) + 1 - \delta\right)}\right) \end{cases}$$

This system of equations describes the dynamics of the model, given the steady rate of human capital. It is a two-dimensional dynamical system in the variables consumption c_t and capital k_t .

6 Equilibria and stability

Let $\phi(c_t, k_t) = \dfrac{(1-g(\alpha))\left[f\left(\dfrac{k_t}{(1-g(\alpha^*))}\right) + (1-\gamma) - (1+\alpha)(1+n)\right] + k_t(1-\delta) - c_t}{(1+\alpha)(1+n)}$

and $\psi(c_t, k_t) = u'^{-1}\left(\dfrac{u'(c_t)(1+n)(1+\alpha)}{\beta\left(f'(\dfrac{k_t}{(1-g(\alpha))}) + 1 - \delta\right)}\right)$. Hence, the system can be rewritten as:

$$\begin{cases} k_{t+1} = \psi(c_t, k_t) \\ c_{t+1} = \phi(c_t, k_t) \end{cases} \tag{19}$$

The equilibria of the model are the constant solutions of the system:

$$\begin{cases} k = \psi(c, k) \\ c = \phi(c, k) \end{cases} \tag{20}$$

Lemma 3. *The system (19) admits a constant solution* (c^*, k^*)

Proof. From equation (18) note that by considering a constant solutions (i.e. $k_t = k$, $c_t = c$, $\forall\ t$), is obtained:

$$f'\left(\frac{k}{(1-g(\alpha))}\right) + 1 - \delta = \frac{(1+\alpha)(1+n)}{\beta}, \tag{21}$$

given the constants values of the parametres and notice that f' is strictly decrease, the only value for k is obtained. Substituting said value in (13) leads to obtaining c, the equilibrium value for the consumption path. Note that k depends on n, α, β, δ, f and g while c depends on n, α, β, δ, γ, f and g. □

Equation (21) require that the steady-state capital-labor ratio does not depend on household preferences except via the discount factor. Moreover, the techonology of production, the depreciation rate, the population growth, the discount factor and the optimal rate of human capital acummulation fully characterize the steady-state capital-labor ratio.

Recalling that $f'\left(\frac{k}{(1-g(\alpha))}\right) + 1 - \delta = 1 + r$, **the gross market interest rate, equation (21) states that at the steady-state point, the interest rate must equalize the product of the human capital rate, population rate and subjective discount rate.**

At this point, it is possible to state that for an interval of α, **at every point of time there exists an unique eqlibrium with some positive fraction of the labor force engaged in education.**

Next in line is to determine its type of equilibrium. For that, attempt to use the Hartman-Grobman theorem.

Lemma 4. *Depending on the values of the parameters* n,β,δ α, γ, *the optimal solution* α *and the functions* f *and* g, *the equilibrium* (c, k) *is a saddle point or a repeller.*

Proof. As it intends to linearize the system (19) is needed to calculate the Jacobian matrix for:

$$\Phi(c, k) = (\phi(c, k), \psi(c, k))$$

and see that its eigenvalues have non zero real parts.

Lets compute the coefficients of the Jacobian matrix for Φ in (c, k):

$$\left.\frac{\partial \phi}{\partial c_t}\right|_{(c,k)} = \frac{-1}{(1+\alpha)(1+n)} := -A$$

$$\left.\frac{\partial \phi}{\partial k_t}\right|_{(c,k)} = \frac{1}{\beta} \quad \text{(this comes from equation (21))}$$

$$\left.\frac{\partial \psi}{\partial c_t}\right|_{(c,k)} = 1$$

$$\left.\frac{\partial \psi}{\partial k_t}\right|_{(c,k)} = -\beta AB$$

where $B = \dfrac{1}{1-g(\alpha)} f''\left(\dfrac{k}{(1-g(\alpha))}\right)\dfrac{u'(c)}{u''(c)}$

So the Jacobian matrix is:

$$J_{(c,k)}\Phi = \begin{pmatrix} -A & \frac{1}{\beta} \\ 1 & -\beta AB \end{pmatrix}$$

Notice that this matrix has a characteristic polynomial:

$$P(\lambda) = \lambda^2 + \lambda A\,(1+\beta B) + \left(\beta A^2 B - \frac{1}{\beta}\right).$$

As $A\,(1+\beta B) \neq 0$, all eigenvalues must have non-zero real part. So, Hartman-Grobman theorem applies.

The eigenvalues are:

$$\lambda_{1,2} = \frac{-A\,(1+\beta B)}{2} \pm \sqrt{\frac{A^2\,(1-\beta B)^2}{4} + \frac{1}{\beta}}$$

As $\dfrac{A^2\,(1-\beta B)^2}{4} + \dfrac{1}{\beta}$ is non negative, both eigenvalues are real.
Also, as $\dfrac{A\,(1+\beta B)}{2} \geq 0$, at least one of them is negative.
Let $\lambda_1 = \dfrac{-A\,(1+\beta B)}{2} - \sqrt{\dfrac{A^2\,(1-\beta B)^2}{4} + \dfrac{1}{\beta}}$, the negative eigenvalue.

Computing $|\lambda_1|$, the condition $|\lambda_1| > 1$ is equivalent to

$$(-A^2 + A)(\beta B) + \frac{1}{\beta} > 1 - A \tag{22}$$

As $0 < A < 1$, $(-A^2 + A)(\beta B) > 0$ and $1 - A < 1$. Noticing that $\frac{1}{\beta} > 1$, the inequality (22) is obtained.
This implies that the equilibrium can not be stable. But to determine if it is a saddle point or a repeller, it is needed to see wheather $|\lambda_2|$ is less or greater than 1.
As $P(-A) < 0$ and $P(\frac{1}{\sqrt{\beta}}) > 0$, $\lambda_2 \in \left(-A, \frac{1}{\sqrt{\beta}}\right)$.
Lets analize $P(0) = \beta A^2 B - \frac{1}{\beta}$. If $P(0) \leq 0$, then $\lambda_2 \in (-A, 0]$, with wich $|\lambda_2| < 1$ and the equilibrium is a saddle point.
Finally, if $P(0) > 0$, then $\lambda_2 \in \left(0, \frac{1}{\sqrt{\beta}}\right)$, so it is positive but could be either greater or less than one.
Similarly as before, $|\lambda_2| < 1$ is equivalent to

$$(-A^2 - A)(\beta B) + \frac{1}{\beta} < 1 + A \tag{23}$$

Analizing equation (23), it is seen that for values $\beta \simeq 1$ the equation is obteined, while for values $\beta \simeq 0$ it is not. So, for the fist case the equilibrium is a saddle point, while in the second case is a repeller.
Sumarizing:

- Linearizing near the equilibrium gives place to two real eigenvalues, λ_1 and λ_2, with $|\lambda_1| > 1$
- If $P(0) \geq 0$, $|\lambda_2| < 1$, so the equilibrium is a **saddle point**
- If $P(0) < 0$, then
  - If $\beta \simeq 1$, $|\lambda_2| < 1$ and the equilibrium is a **saddle point**
  - If $\beta \simeq 0$, $|\lambda_2| > 1$ and the equilibrium is a **repeller**

□

YO AQUI ARRIBA NO PUEDO INTERVENIR CON EL SW; LO DEJO PARA HACER NOTACIONES EN EL PDF Y SE LOS PASO, SIGO ACA ABAJO CON ALGUNAS OBSERVACIONES

- The previous proposition shows that the steady state $\left(\hat{c}, \hat{k}\right)$ is a saddle point for a continuoum of values of β, independently on the values of the other parameters. In this case, the stable transitional path is a one dimensional locus. From here on, we refer to this case, when the equilibrum point of the model is s saddle point.

- The eigenvalue that gives stability tp the system is λ_2 (where $|\lambda_2| < 1$) and, depending on the sign of this value one can have monotonic or oscillatory convergence.

- The equilibrium point levels depend on the following parameters: n,α, β, γ and δ.

- The transitional dynamics around the steady state $\left(\hat{c}, \hat{k}\right)$ can be quantified by using the linearization of the dynamical system (??). In the case when the equilibrium is a saddle point, the matrix $J_G\left(\hat{c}, \hat{k}\right)$ representing the linearization of system (??) has one positive eigenvalue and one negative eigenvalue. Then, there is a straight line of solutions that tend away from the equilibrium and a one dimensional curve of solutions that tend toward the equilibrium as time increases. All other solutions will eventually move away from the equilibrium. This give us the local dynamics of system (??) near the equilibrium $\left(\hat{c}, \hat{k}\right)$: the model exhibits saddle-point stability. Then the dynamic equilibrium follows a stable saddle point expressing the equilibrium $\hat{c}$ as a function of $\hat{k}$. This relation is called the policy function (see [4]).

- The speed of transitional dynamics. The speed of convergence depends on the parameters of technology and preferences and can be computed from the matrix $J_G\left(\hat{c}, \hat{k}\right)$. The eigenvalue λ_1 of $J_G\left(\hat{c}, \hat{k}, \hat{L}\right)$ ($|\lambda_1| > 1$) is excluded from the analysis to ensure the convergence to the equilibrium. The eigenvalue λ_2 ($|\lambda_2| < 1$) is the analogous to the convergence coefficient in standard growth model and corresponds to the negative eigenvalue for the Razin model in continuous time. This eigenvalue depends on parameters of technology and preferencesand corresponds to the source of convergence. The linearization of each stable transition path to the steady state of the system takes the form
$$\begin{cases} c_t = \hat{c} + x\lambda^t \\ k_t = \hat{k} + y\lambda^t \end{cases} \tag{24}$$
where x and y depends on the coefficients of $J_G\left(\hat{c}, \hat{k}\right)$. Then the speed of convergence of consumption and capital depends on eigenvalue λ.

Extraido de Razin (pag. 13): According to our assumptions the term in the brackets on the right hand side of (38) is positive. Therefore, the long run equilibrium of the economy is stable. (Note that in the case of a complete specialization in work the stability of a long run equilibrium is also guaranteed). Therefore, the rate of interest is negatively

related to the Capital Ratio. This implies that the rate of technical progress due to investment in human capital is an increasing function of the Capital Ratio. We might say that the higher the stage of development the higher the rate of technical progress.
We describe in the following diagram, in Figure I, the market paths for the rates of investment in the two forms of capital, for our economy. This picture summarizes our view of how technical progress is related to the other economic variables. In particular, as our model of technical progress implies, the factor which determines the rate of augmentation of the labor productivity, is the Capital Ratio.

7 Conclusions

It is commonly believed that the qualitative implications of an economic dynamic model do not depend on whether time is formulated as a continuous variable or a discrete variable. The choice between continuous-time and discrete-time formulations is usually arbitrary and depends mostly on technical convenience. But the choice of time as a discrete or continuous variable may radically affect the stability of equilibrium in a dynamical economic model. In particular, this paper shows that the dynamics of the Razin model differs dramatically if we represent time as a discrete or a continuous variable. In continuous time, the steady state of the model is locally stable path with monotonic convergence. However, in the discrete time model the steady state may be unstable or saddle path stable with monotonic or oscillatory convergence. The choice between the two types of model is therefore not arbitrary. The diversity of these results highlights the importance of the modeling of time in economic dynamic models.

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