

# Bank's equity-asset ratio bounds under exchange rate risk: a tool for stress testing

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**Abstract** A large fraction of bank's assets and debts in developing countries are usually denominated in foreign currencies (e.g., US dollars, euros). Therefore, fluctuations of the exchange rate is likely to affect their financial wealth. The equity-asset ratio is a usual measure of bank's financial strength. As we will see, this ratio is very sensitive to exchange rate movements. In this paper, we establish some bounds of the expectation of the equity-asset ratio considering limited information of the underlying stochastic exchange rate. Specially, we are interested in finding the lower bound of the expectation of this ratio, that is knowing the worst case scenario of bank financial wealth. We also characterize the conditions under which the equity-ratio remains unchanged considering both direct and indirect effects of exchange rate movements. Our results could shed some light on the complex world of financial regulation and serve regulators as an additional tool for stress testing.

*Keywords and phrases:* Exchange rate risk; stress test; equity-asset ratio; upper bounds; lower bounds; mixture distributions

# 1 Introduction

In dollarized economies, the dollar-local currency exchange rate fluctuations could affect banks financial wealth. Hence, the banks are exposed to an exchange rate risk. The equity-asset ratio or simple equity ratio is an important indicator of banks' financial wealth (Basel Committee on Banking Supervision, 2014; Estrella et al., 2000). More specifically, it measures the proportion of total assets that are financed by stockholders. A positive equity ratio indicates that the bank can repay all its debts with the realization, at face value, of all its assets. On the other hand, a negative equity ratio indicates that the bank cannot repay all its debts with the face value of all its assets, which is a clear indicator of financial distress.

For this reason, Central Banks have strict norms of capital requirement adequacy or leverage ratio rules (Bank for International Settlements, 2010, 2014; Jarrow, 2013a, 2013b; Ong, 2014). Usually, these norms obly banks to maintain a minimum level of the equity ratio both on and off balance sheet items. Additionally, some countries impose limits or restrictions on foreign exchange open positions . The final purpose of these requirements and restrictions is to protect depositors from bank potential losses and thus maintaining the confidence in the financial system (Blum, 2007; Cayazzo et al., 2005; Dermine, 2015; Varotto, 2014).

How could exchange rate changes affect the equity ratio?

There are three main components of this dependence (Cihák, 2014): First, there is a *direct effect* that relates net open positions in foreign currency and the exchange rate. For instance, if there is a long position in the foreign currency, that is the total foreign assets exceeds the total foreign liabilities and the foreign currency appreciates, then the bank will experience a gain for having a positive open position. On the other hand, if there is a short position in the foreign currency, that is the total foreign liabilities exceed the total foreign assets and the foreign currency appreciates, then the bank will experience a loss for having a negative open position. Opposite results hold, in both cases, if the foreign currency depreciates. Second, there is also an *indirect effect* of exchange rate movements. Some of bank's assets are denominated in foreign currency and they could alter their value when the exchange rate changes. For instance, if the local currency depreciates then national borrowers with debts in the foreign-currency, who receives most of their income in local currency, could found difficult to repay their loans. Finally, there is a *liquidity effect* that

results from liquidity mismatches in the inflows and outflows in foreign currency.

Our goal is to study the behavior through finding bounds of the expectation of the equity ratio. We also assume a framework of uncertainty. That is, we find bounds on the equity ratio with limited information about the exchange rate distribution. More specifically, we establish some lower bound of this expectation, therefore we are considering the expectation of the equity ratio of the worst distribution of the exchange rate. This is a common approach used in inventory management (cf. e.g., Gallego & Moon, 1993). Additionally, we establish the conditions under which the expected equity ratio remains unchanged considering both the *direct and indirect effect*. Also, we study the comparative statics of these bounds with respect to the mean and the variance of the exchange rate. We finally establish some bounds when the foreign loans is a discontinuous function of the exchange rate. This assumption allows us to consider jumps on the exchange rate a fact that as has been empirically observed (cf., Jorion, 1988).

Our results could help financial regulators of the financial system as an additional tool for stress testing (for a complete guide of stress testing we refer to Ong, 2014). There several advantages of this tool: i) The bounds are calculated using only accounting data, which is easy to obtain on a monthly basis, ii) they are easy to calculate, ii) they do not require to know the exchange rate distribution. The bounds can be calculated simply knowing the support and the mean of the exchange rate distribution iii) the bounds can be find even with jumps on the exchange rate distribution, iv) the bounds are robust, in the sense that most of the bounds are optimal -given the limited information about the exchange rate distribution and v) the lower bounds serve the regulator of an idea of the bank's financial wealth worst scenario.

The remainder of this paper continues as follows. In Section 2 we present the framework of our model and derive the functional form of the expected equity-ratio. Section 3 presents the main results of our work. First, we establish bounds of the expectation of the equity-ratio and analyze its behavior with both direct and indirect effects of exchange rate movements. Second, comparative statics of these bounds are studied as well. Finally, we propose a loan foreign currency functional form and find lower bounds when the exchange rate behave as a mixture distribution. We finish the paper with the concluding remarks.

## 2 The basic model

In this section we present the set up of the model. We consider a simplified bank balance sheet with loans, deposits and equity denominated in national currency and a single foreign currency. The balance sheet is summarized as follows

Assets	Liabilities+Equity
Loans in local currency	Deposits in local currency
Loans in foreign currency	Deposits in foreign currency
	Equity.

Therefore, we can express the total assets  $A$  as follows:

$$A(x) = L_n + xL(x),$$

where  $x \in R_+$  which is the value that takes the stochastic exchange rate expressed in units of national currency per unit of foreign currency,  $L_n \in R_+$  are the local currency loans and  $L(x)$  are the foreign currency loans expressed as a function of the exchange rate . We assume that the exchange rate does not affect the value of the assets denominated in local currency.

In the same manner, for deposits we have

$$D(x) = D_n + xD_e,$$

where  $D_n \in R_+$  stands for deposits in local currency and  $D_e \in R_+$  stands for deposits in foreign currency. We are also assuming that the exchange rate does not affect the values of the deposits denominated in either foreign or local currency.

By the balance sheet identity, we have that bank's equity  $K$  is equal to

$$K(x) = A(x) - D(x).$$

Then the equity ratio is defined by the ratio between the total capital and total assets:

$$k(x) = \frac{K(x)}{A(x)} = 1 - \frac{D(x)}{A(x)}. \quad (1)$$

After taking expectation in both sides of (1), we arrive at

$$\mathbf{E}[k(X)] = 1 - \mathbf{E}\left[\frac{D_n + XD_e}{L_n + XL(X)}\right], \quad (2)$$

where  $X$  is the stochastic exchange rate. Equation (2) gives us the functional form that we shall study in the rest of the paper.

## 3 Main results

### 3.1 Expectation of the ratio of two functions

We observe that the expected equity-ratio (2) involves the expectation of the ratio of two functions of a random variable. This is a common problem that arises in applied mathematics. As pointed out by many researchers, the expectation of ratio does not necessarily is equal to the ratio of expectations (cf. e.g., Frishman, 1975; Heijmans, 1999). To clarify this, let

$$\mathbf{E} \left[ \frac{f(X)}{g(X)} \right],$$

where  $f$  and  $g$  are two real functions and  $X$  is a random variable.

Recalling that

$$\mathbf{Cov} \left[ g(X), \frac{f(X)}{g(X)} \right] = \mathbf{E} [f(X)] - \mathbf{E} [g(X)] \mathbf{E} \left[ \frac{f(X)}{g(X)} \right], \quad (3)$$

and after rearranging terms in equation (3) we arrive at

$$\mathbf{E} \left[ \frac{f(X)}{g(X)} \right] = \frac{\mathbf{E} [f(X)]}{\mathbf{E} [g(X)]} - \frac{\mathbf{Cov} [g(X), f(X)/g(X)]}{\mathbf{E} [g(X)]}, \quad (4)$$

provided that all the terms exist.

As we deduce from equation (4)

$$\mathbf{E} \left[ \frac{f(X)}{g(X)} \right] = \frac{\mathbf{E} [f(X)]}{\mathbf{E} [g(X)]} \text{ if and only if } \mathbf{Cov} [g(X), f(X)/g(X)] = 0,$$

(see Heijmans, 1999).

Chiang (1966), Fleiss (1966) and Gurland (1967) establish some inequalities of the expectation of the ratio and the ratio of expectations. In the next proposition, we extend the results of these works and find new inequalities that will be useful to establish bounds on the expectation of the equity ratio.

**Proposition 1** *Suppose  $f$  is a differentiable function. Assume  $g$  is a positive, differentiable and increasing function. We have the following assertions:*

1. *If  $f'(x)g(x) > g'(x)f(x)$  for all  $x$  then  $\mathbf{E} \left[ \frac{f(X)}{g(X)} \right] < \frac{\mathbf{E}[f(X)]}{\mathbf{E}[g(X)]}$ ,*

2. if  $f'(x)g(x) = g'(x)f(x)$  for all  $x$  then  $\mathbf{E} \left[ \frac{f(X)}{g(X)} \right] = \frac{\mathbf{E}[f(X)]}{\mathbf{E}[g(X)]}$ , and
3. if  $f'(x)g(x) < g'(x)f(x)$  for all  $x$  then  $\mathbf{E} \left[ \frac{f(X)}{g(X)} \right] > \frac{\mathbf{E}[f(X)]}{\mathbf{E}[g(X)]}$ .

**Proof.** We prove only the first case, the other cases can be proved similarly. Let  $h(x) = f(x)/g(x)$ , then  $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ . By assumption  $f'(x)g(x) > g'(x)f(x)$  implies that  $h'(x) > 0$ . Indeed, since  $h$  is an increasing function together with  $g(x) > 0$  -invoking Chebyshev's inequality (see Egozcue et al., 2009)- implies that

$$\mathbf{Cov} [g(X), f(X)/g(X)] \geq 0.$$

Using now equation (4) the first assertion is easily deduced. ■

### 3.2 Expected equity ratio bounds considering only the direct effect

In this subsection, we shall begin with the simplest banking model assuming that the loans in foreign currency are independent of the exchange, that is  $L(x) = L_e \in R_+$ . Thus, we are considering only the direct effect of the exchange rate movements on the bank's balance sheet.

**Proposition 2** 1. If  $D_e L_n < D_n L_e$  then

$$\mathbf{E} [k(X)] < \frac{\mathbf{E} [K(X)]}{\mathbf{E} [A(X)]} = 1 - \frac{\mathbf{E} [D(X)]}{\mathbf{E} [A(X)]}.$$

2. If  $D_e L_n = D_n L_e$  then

$$\mathbf{E} [k(X)] = \frac{\mathbf{E} [K(X)]}{\mathbf{E} [A(X)]} = 1 - \frac{\mathbf{E} [D(X)]}{\mathbf{E} [A(X)]}.$$

3. If  $D_e L_n > D_n L_e$  then

$$\mathbf{E} [k(X)] > \frac{\mathbf{E} [K(X)]}{\mathbf{E} [A(X)]} = 1 - \frac{\mathbf{E} [D(X)]}{\mathbf{E} [A(X)]}.$$

**Proof.** We only prove (1), the other case can be proved in the same way. The equity ratio is equal to

$$k(x) = \frac{K(x)}{A(x)}.$$

Since  $A(x) = L_n + xL_e$  and  $K(x) = L_n - D_n + x(L_e - D_e)$ , we have

$$K'(x)A(x) - K(x)A'(x) = D_nL_e - D_eL_n > 0.$$

Thus, using the first case of Proposition 1, implies that

$$\mathbf{E}[k(X)] < \frac{\mathbf{E}[K(X)]}{\mathbf{E}[A(X)]}.$$

■

Some additional lower and upper bounds of  $\mathbf{E}[k(X)]$  can be derived from the assumptions of Proposition 2. For example, the condition  $D_eL_n < D_nL_eD$  implies that  $k$  is strictly increasing. If we assume that the support of  $X$  is  $[0, \infty)$  then

$$k(0) \leq \mathbf{E}[k(X)] \leq \lim_{x \rightarrow \infty} k(x).$$

We have

$$k(0) = 1 - \frac{D_n}{L_n}$$

and for the  $\lim_{x \rightarrow \infty} k(x)$  we have an indetermination of  $\frac{\infty}{\infty}$ , using L'Hopital rule we arrive at

$$\lim_{x \rightarrow \infty} k(x) = 1 - \lim_{x \rightarrow \infty} \frac{D'(x)}{A'(x)} = 1 - \frac{D_e}{L_e}.$$

Hence

$$1 - \frac{D_n}{L_n} \leq \mathbf{E}[k(X)] \leq 1 - \frac{D_e}{L_e} \tag{5}$$

If  $D_eL_n > D_nL_eD$  inequalities in (5) are reverted.

The following corollary shows how the expectation of the equity ratio relates to the mean of  $X$ .

**Corollary 1** *Let  $X$  be random variable with mean equal to  $\mu$ .*

1. *If  $D_eL_n < D_nL_e$  then  $\mathbf{E}[k(X)] \leq k(\mu)$ .*
2. *If  $D_eL_n = D_nL_e$  then  $\mathbf{E}[k(X)] = k(\mu)$ ,*
3. *If  $D_eL_n > D_nL_e$  then  $\mathbf{E}[k(X)] \geq k(\mu)$ .*

To gain some intuition about Proposition 2 and Corollary 1, let's look at the second case. In this regard, Cayasso et al. (2005) and Pena (2006) show that if the following equalities are fulfilled

$$L_n - D_n = k(x)L_n \text{ and } L_e - D_e = k(x)L_e, \text{ for all } x, \quad (6)$$

then  $k(X)$  is a degenerated random variable. We pause here, to define currency open positions. Formally, the open positions of the local and foreign currency are  $P_n = L_n - D_n$  and  $P_e = L_e - D_e$ , respectively. The open exchange position is:

- i) long if  $P_i > 0$ ,
  - ii) balanced if  $P_i = 0$  and
  - iii) short if  $P_i < 0$ ,
- for  $i = e, n$ .

Note that the equations (6) are equivalent to

$$k(X) = \frac{P_n}{L_n} = \frac{P_e}{L_e},$$

which is also equivalent to

$$D_e L_n = D_n L_e. \quad (7)$$

We see that the ratio of the proportion of debts in foreign currency and the proportion in local currency plays a fundamental role in this relationship. Usually, balanced foreign position is considered to be a risk-free position, however, as Corollary 1 shows, foreign exchange movements could have a significant impact on the equity-ratio, even with a foreign currency balanced open position. More specifically, observe that

$$\mathbf{E}[k(X)] = \frac{P_n + \mathbf{E}[X] P_e}{L_n + \mathbf{E}[X L(X)]}.$$

When  $L(X) = L_e$  and  $\frac{P_n}{L_n} = \frac{P_e}{L_e}$  then

$$\mathbf{E}[k(X)] = \frac{P_n + \mathbf{E}[X] P_e}{L_n + L_e \mathbf{E}[X]} = \frac{P_n \left(1 + \frac{P_e}{P_n} \mathbf{E}[X]\right)}{L_n \left(1 + \frac{L_e}{L_n} \mathbf{E}[X]\right)} = \frac{P_n}{L_n} = k(\mu),$$

where  $\mathbf{E}[X] = \mu$ . Therefore, if the open positions relatively to the loans in their respective currency are equal, then the expectation of the equity ratio is equal to the ratio of the expectation. In particular, if we have balanced open positions in local and foreign currency then  $\mathbf{E}[k(X)] = 0$

### 3.2.1 Some bounds using only the mean and support of $X$ .

We can improve the bounds studied in Subsection 3.2 if we have some additional information about the stochastic exchange rate  $X$ . We start here with the simplest case which is finding bounds when we only know the mean and support of  $X$ .

Edmundson (1957) and Madansky (1959) the following inequality: Consider a random variable  $X$  with mean  $\mu$  and support on  $[a, b]$ . Let  $f$  be a convex function on  $[a, b]$ . Then

$$\mathbf{E}[f(X)] \leq \frac{(b - \mu) f(a) + (\mu - a) f(b)}{b - a}. \quad (8)$$

If  $f$  is concave then inequality (8) is reversed.

This is an important result that help us to find upper or lower bounds of the equity-ratio with the limited information of knowing only the mean and the support of the exchange rate distribution as we do next.

**Proposition 3** *Let  $X$  be a random variable that takes values in  $[a, b]$  and has expected value  $\mu$ . Then:*

1. *if  $D_e L_n < D_n L_e$  then*

$$\mathbf{E}[k(X)] \geq \left(\frac{b - \mu}{b - a}\right) k(a) + \left(\frac{\mu - a}{b - a}\right) k(b) = EM, \quad (9)$$

2. *if  $D_e L_n > D_n L_e$  then*

$$\mathbf{E}[k(X)] \leq \left(\frac{b - \mu}{b - a}\right) k(a) + \left(\frac{\mu - a}{b - a}\right) k(b).$$

**Proof.** The first case. Note that

$$k''(x) = \frac{\delta^2 k}{\delta x^2} = -2 \frac{(D_n L_e - D_e L_n)}{A^3(e)} L_e,$$

since  $D_n L_e - D_e L_n > 0$ , we have

$$k''(x) < 0,$$

and hence  $k$  is a concave function. Now, using inequality (8), but now applied to concave functions, we arrive at

$$\mathbf{E}[k(X)] \geq \left(\frac{b - \mu}{b - a}\right) k(a) + \left(\frac{\mu - a}{b - a}\right) k(b).$$

For the second case, if  $D_e L_n > D_n L_e$  then  $k(x)$  is a convex function and using inequality (8) we have

$$\mathbf{E}[k(X)] \leq \left(\frac{b - \mu}{b - a}\right) k(a) + \left(\frac{\mu - a}{b - a}\right) k(b).$$

■

### 3.2.2 Some bounds using the mean and variance of $X$ .

In the same manner, if we know some additional moments of  $X$  we can improve Edmundson-Madansky bound. Indeed, Willassen (1981) derive the following inequality considering a random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$  and  $f$  an increasing and concave function on  $[0, +\infty)$ :

$$\mathbf{E}[f(X)] \geq \frac{\mu^2}{\mu^2 + \sigma^2} f\left(\mu + \frac{\sigma^2}{\mu}\right). \quad (10)$$

In general, this optimal bound cannot be improved even if the third moment of  $X$  is known (Willassen, 1990).

Indeed, we note that Willassen's bound can be extended for a random variable  $X$  with support on  $[a, +\infty)$ . To show this, let's define a function  $U$  with domain in  $[a, +\infty)$  with  $U(a) = 0$ .

Suppose that  $X$  has mean  $\mu$  and variance  $\sigma^2$ . Let  $Y = X - a$  then  $Y$  has support on  $[0, +\infty)$  and its mean is  $\mu' = \mu - a$  and its variance continues to be  $\sigma^2$ . Therefore, we have

$$\mathbf{E}[f(X)] = \mathbf{E}[f(Y + a)] = \mathbf{E}[v(Y)],$$

with  $v(y) = f(y + a)$  and  $v(0) = u(a) = 0$ . Then we can use bound (10) and conclude that:

$$\mathbf{E}[v(Y)] \geq \frac{\mu'^2}{\sigma^2 + \mu'^2} v\left(\mu' + \frac{\sigma^2}{\mu'}\right).$$

After some algebra, we obtain that

$$\mathbf{E}[f(X)] \geq \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2} f\left(\mu + \frac{\sigma^2}{\mu - a}\right). \quad (11)$$

The bound for  $\mathbf{E}[v(Y)]$  is sharp since it is obtained by the following two point distribution:

$$\mathbf{P}[Y = 0] = \frac{\sigma^2}{\mu'^2 + \sigma^2} \text{ and } \mathbf{P}\left[Y = \mu' + \frac{\sigma^2}{\mu'}\right] = \frac{\mu'^2}{\mu'^2 + \sigma^2}.$$

Similarly for  $X$  the bound is sharp in the following distribution:

$$\mathbf{P}[X = a] = \frac{\sigma^2}{(\mu - a)^2 + \sigma^2} \text{ and } \mathbf{P}\left[X = \mu + \frac{\sigma^2}{\mu - a}\right] = \frac{(\mu - a)^2}{(\mu - a)^2 + \sigma^2}. \quad (12)$$

Furthermore, the bound is still optimal even if the support of  $X$  instead of  $[a, +\infty)$  is  $[a, b]$ . To verify this, we simply need to check that the two point distribution in (12) has support on  $[a, b]$ . Note that this holds if

$$\mu + \frac{\sigma^2}{\mu - a} \leq b$$

which is equivalent to:

$$\sigma^2 \leq (b - \mu)(\mu - a), \quad (13)$$

which holds by a well known variance inequality (cf., Muilwijk, 1966). The variance's bound (13) show us that we can also used bound (10) even if we do not know the variance of  $X$  -as long as we do know its mean and support.

We use bound (11) to find lower bounds of the equity ratio in the next proposition.

**Proposition 4** *Suppose that  $X$  has support on  $[a, b]$  with mean and variance  $\mu, \sigma^2$  respectively.*

1. *If  $D_e L_n < D_n L_e$  then*

$$\mathbf{E}[k(X)] \geq \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2} k\left(\mu + \frac{\sigma^2}{\mu - a}\right) + \frac{\sigma^2}{\sigma^2 + (\mu - a)^2} k(a) = W, \quad (14)$$

and

2. *if  $D_e L_n > D_n L_e$  then  $\mathbf{E}[k(X)] \geq k(\mu)$ .*

**Proof.** The proof of the first case. Note that if  $D_e L_n < D_n L_e$  then  $k$  is increasing and concave and its third derivative is

$$k'''(x) = \frac{6(L_e D_n - L_n D_e)L_e^2}{(L_n + xL_e)^4} > 0.$$

So far, the only assumption that departs to from the assumptions used to derive bound (10) is the requirement that  $k(0) = 0$ , which is not crucial, as we show next. Let  $t(x) = k(x) - k(a)$ . Hence,  $t$  is increasing, concave,  $t''' > 0$  and  $t(a) = 0$ , therefore we are under the same assumptions as in Willassen's framework and we can use bound (11) . Hence,

$$\begin{aligned} \mathbf{E}[k(X)] &= \mathbf{E}[t(X)] + k(a) \\ &\geq \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2} t\left(\mu + \frac{\sigma^2}{\mu - a}\right) + k(a) \\ &= \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2} k\left(\mu + \frac{\sigma^2}{\mu - a}\right) - \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2} k(a) + k(a) \\ &= \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2} k\left(\mu + \frac{\sigma^2}{\mu - a}\right) + \frac{\sigma^2}{\sigma^2 + (\mu - a)^2} k(a). \end{aligned}$$

To prove the second case, note that if  $D_e L_n > D_n L_e$  then  $k$  is a convex function-invoking Jensen's inequality we arrive at the assertion. ■

**Remark 1** *The bounds in Proposition 4 are sharp. In the first case, the worst distribution function is*

$$\mathbf{P}[X = a] = \frac{\sigma^2}{\sigma^2 + (\mu - a)^2} \text{ and } \mathbf{P}\left[X = \mu + \frac{\sigma^2}{\mu - a}\right] = \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2}.$$

*Straightforward calculation gives an expected equity-ratio equal to the first lower bound. In the second case, simply setting*

$$\mathbf{P}(X = \mu) = 1,$$

*yields the lower bound.*

Another bound apart from  $W$  that uses higher moments of  $X$  is due to Dokov and Morton (2002). (). By invoking Bernstein's polynomial they obtain the following bound when the mean  $\mu$  and variance  $\sigma^2$  of  $X$  are known:

$$DM = \frac{1}{(b-a)^2} \left[ f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) \right] \sigma^2 + \left[ \left(\frac{b-\mu}{b-a}\right)^2 f(a) + \frac{2(\mu-a)(b-\mu)}{(b-a)^2} f\left(\frac{a+b}{2}\right) + \left(\frac{\mu-a}{b-a}\right)^2 f(b) \right]. \quad (15)$$

**Remark 2** *However, as bound (14) is optimal then bound (15) is, in general, not optimal. Nevertheless, we consider interesting to present it and compare some of its properties with those of bounds  $EM$  and  $W$ .*

The next corollary show the relationship between bounds (??), (14) and (15) if  $X$  has the maximum variance.

**Corollary 2** *Under the assumptions of Proposition 4, if  $\sigma^2 = (b - \mu)(\mu - a)$  then*

$$EM = W = DM.$$

### 3.3 Some bounds comparative statics

In the next proposition, we study the behavior of the bounds  $EM, W$  and  $DM$  with the movements of the mean and the variance.

**Proposition 5** *Under the assumptions of Proposition 4 and  $D_e L_n < D_n L_e$  then:*

1.  $\frac{\delta EM}{\delta \mu} > 0$ ,
2.  $\frac{\delta W}{\delta \mu} > 0$ ,
3.  $\frac{\delta DM}{\delta \mu} > 0$ ,
4.  $\frac{\delta EM}{\delta \sigma^2} = 0$ ,
5.  $\frac{\delta W}{\delta \sigma^2} < 0$ ,
6.  $\frac{\delta DM}{\delta \sigma^2} < 0$ .

**Proof.** To prove (1), note that

$$\frac{\delta EM}{\delta \mu} = \frac{k(b) - k(a)}{b - a}.$$

If  $D_e L_n < D_n L_e$  then  $k$  is an increasing function and  $\frac{\delta EM}{\delta \mu} > 0$ . Now, we prove (2). Let  $t = \frac{(\mu - a)^2}{\sigma^2 + (\mu - a)^2}$ . We have:

$$W = tk \left( a + \frac{\sigma}{\sqrt{t(1-t)}} \right) + (1-t)k(a).$$

Now:

$$\frac{dW}{d\mu} = \frac{dW}{dt} \frac{dt}{d\mu}.$$

Since  $\frac{dt}{d\mu} > 0$  we only have to check the sign of  $\frac{dW}{dt}$ . After some algebra we obtain

$$\frac{dW}{dt} = \frac{(D_n L_e - D_e L_n) \sigma \left( a L_e t + 2\sigma L_e \sqrt{t(1-t)} + L_n t \right)}{2\sqrt{t(1-t)}(a L_e + L_n) \left( L_e \left( a\sqrt{t(1-t)} + \sigma \right) + L_n \sqrt{t(1-t)} \right)^2},$$

which is positive since we assume that  $D_e L_n < D_n L_e$ . The proof of (3), after some calculation we arrive at

$$\frac{dDM}{d\mu} = \frac{2(D_n L_e - D_e L_n)(L_e(a + b - \mu) + L_n)}{(a L_e + L_n)(b L_e + L_n)(a L_e + b L_e + 2L_n)},$$

which is also positive. For (4), we simple need to note that  $EM$  is independent of  $\sigma^2$ . Now case (5), with the same change of variable used in the proof of (2) we obtain

$$W = tk \left( a + \frac{\mu - a}{t} \right) + (1-t)k(a)$$

and

$$\frac{dW}{d\sigma^2} = \frac{dW}{dt} \frac{dt}{d\sigma^2}.$$

In this case  $\frac{dt}{d\sigma^2} < 0$  so we must verify that  $\frac{dW}{dt} > 0$ . After some tedious calculation it yields that

$$\frac{dW}{dt} = \frac{(D_n L_e - D_e L_n) L_e (a - \mu)^2}{(a L_e + L_n)(a L_e (t - 1) + L_e \mu + L_n t)^2},$$

which is positive by our assumptions. Finally for (6), note that

$$\frac{\delta DM}{\delta \sigma^2} = \frac{1}{(b - a)^2} \left[ k(a) + k(b) - 2k\left(\frac{a + b}{2}\right) \right],$$

which is negative by invoking Jensen's inequality. ■

In Proposition 5 all the inequalities are reverted if  $D_e L_n > D_n L_e$ . To gain some intuition on these results, first note that bounds increases with  $\mu$  as long as  $D_e L_n < D_n L_e$ . In particular, note that if  $L(x) = L_e$  then  $\mathbf{E}[k(X)] = k(\mu)$ . Furthermore,  $k$  is increasing and concave if  $D_e L_n < D_n L_e$ , thus  $\frac{\delta \mathbf{E}[k(X)]}{\delta \mu} = k'(\mu) > 0$  and the bounds move along  $\mathbf{E}[k(X)]$ .

The next examples illustrates our previous findings.

**Example 1** Assume that the support of the  $X$  is  $[1, 5]$  and  $\mu = 2$ . Then

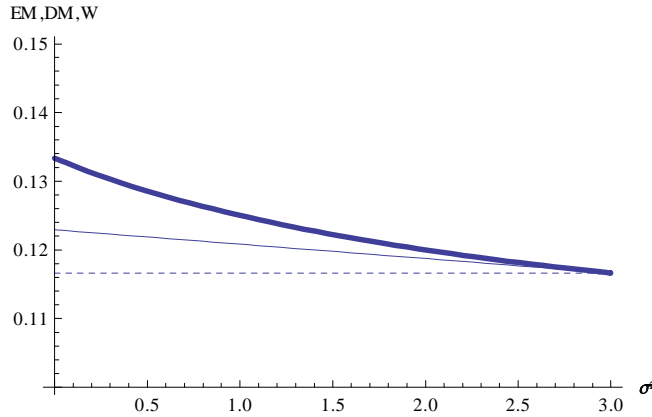


Figure 1:  $L_n = D_n = 10 = L_e = 10$  and  $D_e = 8$ .  $W$ - bold solid line,  $DM$ - solid line and

EM- dashed line.

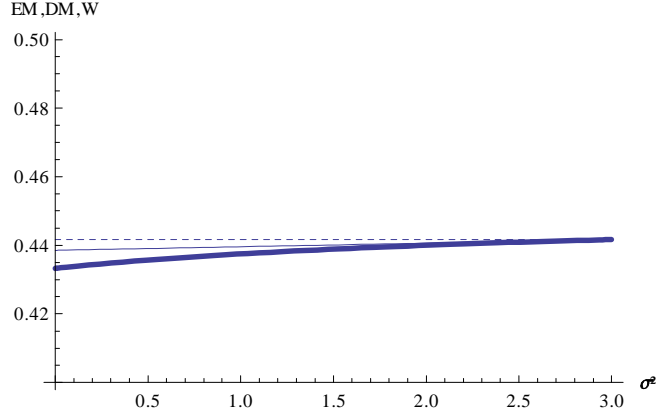


Figure 2:  $L_n = L_e = 10$ ,  $D_n = 5$  and  $D_e = 6$ . W- bold solid line, DM- solid line and EM- dashed line.

### 3.4 Constant equity ratio with both direct and indirect effects

At first glance, it appear that the equity ratio cannot be constant when the foreign loans is not a constant function of the exchange rate, that is, when we consider both the direct and indirect effect in the equity ratio with movements in the exchange rate. However, in the next proposition, we establish the conditions where the expectation of the equity ratio is constant even if the foreign loans are not constant.

**Proposition 6** *The equity ratio is constant if and only if there exist a positive constant  $\theta$  such that*

$$L(x) = \theta + \frac{\gamma\theta - L_n}{x}, \quad (16)$$

where  $\gamma = D_n/D_e$ . Moreover,

$$\mathbf{E} [k(X)] = 1 - \frac{D_e}{\theta}.$$

**Proof.** The equity ratio is constant if and only if

$$k'(x) = \frac{\delta k(x)}{\delta x} = 0,$$

for all  $x$ . This is equivalent to solving the following first order linear differential equation,

$$\frac{\delta L(x)}{\delta x} + \frac{D_n L(x)}{x D(x)} = \frac{D_e L_n}{x D(x)}.$$

Note that this differential equation is of the form

$$y' + P(x)y = Q(x), \quad (17)$$

where  $y' = \frac{\delta L(x)}{\delta x}$ ,  $y = L(x)$ ,  $P(x) = D_n/xD(x)$  and  $Q(x) = D_e L_n/xD(x)$ . The solution of (17) is

$$y(x) = \frac{1}{u(x)} \int u(x)Q(x)dx,$$

where  $u(x) = e^{\int P(x)dx}$ , (see Zill, 2012). Observe that

$$\int P(x)dx = \ln x - \ln(D_n + xD_e).$$

Hence

$$u(x) = \frac{x}{D(x)}$$

and

$$\int u(x)Q(x)dx = -\frac{L_n}{D(x)},$$

Therefore,

$$y(x) = -\frac{L_n}{x} + \frac{\gamma}{u(x)} = -\frac{L_n}{x} + \frac{\gamma D_n + \gamma x D_e}{x} = \frac{\gamma D_n - L_n}{x} + \gamma D_e.$$

for some constant  $\gamma$ . If we set  $\theta = \gamma D_e$ , we arrive at the same expression as in (16).

We provide another much simpler proof. We first prove the sufficient part. Assume that  $k$  is a constant function, that is  $A(x) = \gamma D(x)$  for some constant  $\gamma \in R_+$  and for all  $x \in R_+$ .

Let  $\theta = \gamma D_e$ , hence

$$\begin{aligned} A(x) = \gamma D(x) &\Rightarrow L_n + xL(x) = \frac{\theta}{D_e}(D_n + xD_e) \Rightarrow \\ &\Rightarrow L_n + xL(x) = \theta\gamma + \theta x \Rightarrow L(x) = \theta + \frac{\gamma\theta - L_n}{x}, \end{aligned}$$

where  $\gamma = D_n/D_e$ . Now, the necessity part. Assume that

$$L(x) = \theta + \frac{\gamma\theta - L_n}{x}.$$

Then

$$A(x) = L_n + xL(x) = L_n + x \left( \theta + \frac{\gamma\theta - L_n}{x} \right) = \frac{\theta}{D_e}(D_n + xD_e) = \frac{\theta D(x)}{D_e}.$$

Hence,

$$k(x) = 1 - \frac{D(x)}{A(x)} = 1 - \frac{D_e}{\theta}. \quad (18)$$

This finishes the necessity part. Finally, taking expectation in both sides of (18) we obtain the expectation of the equity ratio assertion. ■

**Remark 3** In particular, if  $\theta = (L_n/D_n) D_e$  then

$$L(x) = \frac{L_n}{D_n} D_e,$$

which is equal to the condition (7).

### 3.5 Expected equity ratio bounds with both direct and indirect effects

The functional form of the loans in (16) has some unlikely realistic properties. For instance, its value tends to infinity as the exchange rate approaches zero. Also, the foreign loans value is a continuous function of the exchange rate. We expect that its value behaves as a discontinuous function of the exchange rate. Moreover, it is reasonable to assume that  $L(x)$  is constant if the the exchange rate does not vary much, and for the reasons exposed in Section 1, decreases as long as the exchange increases unexpectedly (i.e. it has some jumps and the local currency devaluates).

Therefore, we assume the following function form for the foreign loans

$$L(x) = L_e \cdot \mathbf{1}\{x \in [a, b]\} + \lambda L_e \cdot \mathbf{1}\{x \in [c, d]\}, \quad (19)$$

where  $L_e, a, b, c, d$  are positive constants such that  $a \leq b < c \leq d$ ,  $0 \leq \lambda \leq 1$  and  $\mathbf{1}\{x \in A\}$  is the indicator function, which is equal to 1 if  $x \in A$  and equal to zero if  $x \notin A$ . The parameter  $\lambda$  reflects the value of the foreign loans when the exchange rate has jumped to a value in  $[c, d]$ . Thus,  $1 - \lambda$  is the loss of the foreign loan  $L_e$  after the exchange rate is in certain range. Or putting differently, is the proportion of performing loans that passes to default loans after a jump on the exchange rate. This is a similar analysis used to study collateral rules proposed in the literature (cf, Jarrow, 2013a).

In turn, if the foreign loans are independent of the exchange rate then we simply define  $L(x) = L_e \in R_+$ . As a general function, the equity ratio will be equal to

$$k(x) = \begin{cases} 1 - \frac{D_n + x D_e}{L_n + x L_e} & \text{if } x \in [a, b], \\ 1 - \frac{D_n + x D_e}{L_n + \lambda x L_e} & \text{if } x \in [c, d]. \end{cases} \quad (20)$$

The first case of independence holds when  $\lambda = 1$ .

In general, we cannot apply bounds  $W$  and  $DM$  because they require the continuity of  $k$ , which is not the case in (20). To simplify the analysis, we assume that the exchange rate

behaves as a mixture distribution of two random variables (the generalization to  $n$  random variables is straightforward). Consider a random variable  $Y_1$  with support on  $[a, b]$  and a random variable  $Y_2$  with support on  $[c, d]$ . Now, we assume the exchange rate is equal to

$$X = \begin{cases} Y_1 & \text{with probability } p, \\ Y_2 & \text{with probability } 1 - p. \end{cases}$$

This assumption allow us to consider jumps in the exchange rate. It can be proved that its moments are equal to

$$\mathbf{E}[X^n] = p\mathbf{E}[Y_1^n] + (1 - p)\mathbf{E}[Y_2^n] \text{ for all } n \in N,$$

(see, Bean, 2001). However, under certain conditions, we can apply both bounds as we show in the next propositions.

**Proposition 7** *Suppose  $X$  is a mixture distribution of the random variables  $Y_1$  with mean  $\mu_1$  and support on  $[a, b]$  and  $Y_2$  mean  $\mu_2$  and support on  $[c, d]$ .*

1. *If  $L(x)D_n > L_nD_e$  for all  $x \in [a, b] \cup [c, d]$  then*

$$\mathbf{E}[k(X)] \geq p \left( \frac{b - \mu_1}{b - a} k(a) + \frac{\mu_1 - a}{b - a} k(b) \right) + (1 - p) \left( \frac{d - \mu_2}{d - c} k(c) + \frac{\mu_2 - c}{d - c} k(d) \right), \quad (21)$$

2. *If  $L(x)D_n < L_nD_e$  for all  $x$  then*

$$\mathbf{E}[k(X)] \geq pk(\mu_1) + (1 - p)k(\mu_2).$$

In the same manner, we can establish lower bounds for an exchange rate with a mixture distribution using Willassen's bounds.

**Proposition 8** *Suppose  $X$  is a mixture distribution of the random variables  $Y_1$  with mean  $\mu_1$ , variance  $\sigma_1^2$  and support on  $[a, b]$  and  $Y_2$  with mean  $\mu_2$ , variance  $\sigma_2^2$  and support on  $[c, d]$ . If  $L(x)D_n > L_nD_e$  for all  $x \in [a, b] \cup [c, d]$  then*

$$\begin{aligned} \mathbf{E}[k(X)] \geq & p \left( \frac{(\mu_1 - a)^2}{\sigma_1^2 + (\mu_1 - a)^2} k \left( \mu_1 + \frac{\sigma_1^2}{\mu_1 - a} \right) + \frac{\sigma_1^2}{\sigma_1^2 + (\mu_1 - a)^2} k(a) \right) \\ & + (1 - p) \left( \frac{(\mu_2 - c)^2}{\sigma_2^2 + (\mu_2 - c)^2} k \left( \mu_2 + \frac{\sigma_2^2}{\mu_2 - c} \right) + \frac{\sigma_2^2}{\sigma_2^2 + (\mu_2 - c)^2} k(c) \right). \end{aligned} \quad (22)$$

The following examples illustrate the results of Proposition 7 and 8.

**Example 2** Assume that  $L_n = L_e = D_n = 10$  and  $D_e = 8$ . To illustrate an application of Proposition 7 and 8, let's assume that  $Y_1$  has support in  $[0, 1]$  and  $Y_2$  has support in  $[2, 3]$ , with means  $\mu_1 = 0.5$ ,  $\mu_2 = 2.5$ . Notice that  $L(x)D_n > L_n D_e$  if  $\lambda \in (0.8, 1]$ . Then, by bound 22

$$\begin{aligned} \mathbf{E}[k(X)] \geq & p \left( \frac{0.25}{\sigma_1^2 + 0.25} k \left( 0.5 + \frac{\sigma_1^2}{0.5} \right) \right) \\ & + (1 - p) \left( \frac{0.25}{\sigma_2^2 + 0.25} k \left( 2.5 + \frac{\sigma_2^2}{0.5} \right) + \frac{\sigma_2^2}{\sigma_2^2 + 0.25} k(2) \right). \end{aligned}$$

Notice that bound 7 is equal to bound 22 if we have maximum variances, that is  $\sigma_1^2 = \sigma_2^2 = \frac{1}{4}$  and

$$\mathbf{E}[k(X)] \geq \frac{p}{2} k(1) + \frac{(1-p)}{2} (k(2) + k(3)).$$

## 4 Concluding remarks

In this paper, we derive some bounds on the expectation of the bank equity ratio. This is an important indicator of bank's financial wealth and has been used in the financial system to regulate. Our proposal has many advantages of its use. The bounds are simple to calculate since they rely in accounting data, which is available in a monthly basis. The bounds can be calculated with limited information about the underlying exchange rate distribution. Moreover, we allow this distribution to have jumps. Finally, the lower bounds of the equity ratio indicates how bad a bank could be.

This paper can be improved in several directions. First, our static analysis could be extended to a dynamic study. Second, new functional forms of the foreign loans could be proposed. Third, empirical study of these bounds, specially in dollarized economies in crisis, remain as a future task.

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