Assessing Dependence Between Financial Market Indexes Using Conditional Time-Varying Copulas: Applications to Value at Risk (VaR)

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Abstract: In this paper, we analyze the time dynamics of the dependence structure between IBOVESPA (Índice da Bolsa de Valores do Estado de São Paulo) and the following three indexes: FTSE100 (Financial Times and London Stock Exchange Index), IPCMX (Índice de Precios y Cotizaciones da Bolsa Mexicana de Valores) and S&P500 (Standard and Poor 500 Index). We follow Patton’s (2006) conditional copula setting and additionally observe the impact of different copula functions on Value at Risk (VaR) estimation for a naive proposed portfolio. We conclude that the dependence between IBOVESPA and other financial market indexes has intensified from the beginning of 2006. Furthermore in our case the copula form seems not to be relevant for VaR estimation, since all copulas lead to significant VaR estimates. Finally, to identify which copula functions lead to the best fit to the data we apply a goodness-of-fit test based on parametric bootstrap. We find that the best fits are obtained via the time constant t-student and the time-varying Normal copulas.

Keywords: Asymmetric dependence, Time-varying copulas, Value at Risk (VaR), Skewed-t GARCH, Bootstrap test.

JEL Classification: C15, C46, G15.
Introduction

It is recognized in the literature that many economic variables are not normally distributed, mainly when it comes to financial data\(^1\). These variables show heavy-tails, or kurtosis excess, asymmetry and, as recent works suggest, the existence of "asymmetric dependence", where some pairs of variables are more highly correlated during downwards movements than during upwards movements. Furthermore, as observed by Christoffersen (2006), this "asymmetric dependence" varies over time. In face of these findings, some important questions have been raised in the literature:

1. What is the appropriate multivariate distribution to model financial data?
2. What measures of dependence explain properly the types of correlation found in financial data?
3. What is the effect of the assumed dependence structure on risk estimates?

Stylized Facts on asset returns describe some statistical characteristics commonly present in financial time series. Among others we remind the following: (i) heavy tails in the returns unconditional distribution, i.e., returns distributions tends to be leptokurtic (Sancetta and Satchell, 2001); (ii) volatility clustering, which means an alternate presence of high and low volatility clusters over time; (iii) correlation between the volume of traded assets and their volatility; (iv) asymmetry between gain and loss: there is a greater magnitude of losses during decreasing movements than of gains during increasing movements (see Cont, 2001).

Despite these facts, multivariate normality is often assumed in finance when it comes to portfolios analysis (Sancetta, 2004). If one restricts themselves to assuming elliptical distributions (Gaussian, t-student), one deals with measures of linear dependence between the financial returns, which may not completely capture the type of dependence present in the data. Hence there is a need to build multivariate distributions that allow one to model data with different patterns of dispersion, where asymmetry, heavy-tails and other financial returns behavior are accommodated. It is also necessary to consider broader measures of dependence that take into account the behavior of financial returns.

Our approach here is to use copulas to deal with the issues raised above. We analyze the dependence structure between financial assets and additionally compute the Value at Risk (VaR), which is of considerable importance in risk analysis (see for instance Jorion, 1997). The effect of the different copulas on VaR estimation is studied. Furthermore we try to identify the best copula fit via a parametric bootstrap method suggested by Allcroft and Glasbey (2003) in a more general context.

Informally speaking, copulas are functions that connect multivariate distribution functions to their marginal distributions of any dimension. They have all the relevant information about the dependence structure among the variables. This allows greater flexibility in modeling the multivariate distributions and their margins. Firstly, because there are more details in the full set of copula specifications than there are in the traditional multivariate setting by itself. Secondly, the underlying methodology allows joint distributions to be derived from their marginal distributions even when the latter are not normal distributions. And finally, it is possible to separate the characteristics of each marginal from the dependence parameter.

In the copula literature, the issues of dependence structures and multivariate distributions derivations have been studied by Clemen and Reilli (1999), Mendes and Moretti (2002), Embrechts, McNeil and Straumann (2002), Embrechts, Hoeing and Jüri (2003), Hurd, Salmon and Schleicher (2005), among others. In their great majority however, these studies assume that the dependence relations among the variables are constant over time.

More recently we can see the emergence of time-varying conditional copulas. In Mendes (2005), the dependence parameter is conditioned on the position of combined past observations in the unit square to which they belong, i.e., the unit square is divided into several subgroups and then the dependence parameter is estimated for each subgroup. Using Gaussian and t-student copulas, the VaR is calculated for Argentinean and Brazilian financial market indexes. Ignatieva (2005) in turn uses an adaptive approach

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\(^1\)See, for example, Daal, Naka and Yu (2004) and Malevergne and Sornette (2006).
known as local change point detection analysis (LCPD), which is based on the assumption that there is a local time homogeneity, i.e., for every moment in time there is an interval in which the dependence parameter is approximately constant. This approach is used in the analysis of dependence between some pairs of assets comprised in the DAX index (Deutscher AKTIEN-Index). Patton (2006) seems to be the first to extend the theorem of Sklar (1956) on copulas for a conditional version. The author illustrates his approach estimating and evaluating time-varying conditional densities of the exchange rates German mark/dollar and yen/dollar returns. The equation of evolution used by Patton forces the dependence parameters to follow a constrained ARMA (1, 10) process. So a symmetrized Joe-Clayton time-varying copula is generated.

Taking into account these initial considerations, the aim of this study is to measure the dependence structure between IBOVESPA (índice da Bolsa de Valores de São Paulo) and the following three indexes: FTSE100 (Financial Times and London Stock Exchange Index), IPCMX (Índice de Precios y Cotizaciones da Bolsa Mexicana de Valores) and SP500 (Standard and Poor 500 Index). To capture this dependence structure we use four conditional copulas with time-varying parameters: Symmetrized Joe Clayton, Rotated Gumbel, Normal and T-student. The time dynamics of the dependence parameter follow those proposed by Patton (2006). We evaluate the copulas fit via a parametric bootstrap method, as mentioned previously. Therefore, using the approach of conditional densities estimated via copulas, we compute the Value at Risk (VaR) for the portfolios formed by the indexes in question.

The rest of the paper is comprised as follows. In Section 2 we present a summary of the theory of conditional copulas and their estimation via maximum likelihood also describing the parametric forms used in this work. In Section 3 we describe our data, model the indexes margins and fit the parametric copulas along with their dependence parameters. Section 4 evaluates the goodness-of-fit of our copula functions using parametric bootstrap. In Section 5 we analyze the time dynamics of the dependence parameters. Section 6 in its turn brings VaR definition and its estimation via copulas for a particular proposed portfolio constructed for our data. Lastly, some closing comments are presented.

2 Conditional copulas

Copulas are often defined in literature as distribution functions whose marginal distributions are uniform in the interval \([0,1]\). That is, for an n-dimensional vector \(U\) in the unit cube, a copula can be informally defined as

\[
C(u_1,\ldots,u_n) = \Pr(U_1 \leq u_1,\ldots,U_n \leq u_n),
\]

(1)

where \(U_i\) is a random variable with uniform distribution in \([0, 1]\) and \(u_i\) is a realization of \(U_i, i = 1,2,\ldots,n\).

Combining this with the fact that continuous random variables can be transformed into uniform variables by their probability integral transform, copulas can be used to achieve structures of multivariate dependence from the marginal distributions.

2.1 Basic concepts

Formally, we can define a copula as follows.

**Definition 1**: An n-dimensional copula is a function \(C\) with domain \([0,1]^n\), such that:

1. \(C\) is grounded and n-increasing.
2. \(C\) has margins \(C_k, k = 1,2,\ldots,n\), where \(C_k(u) = u\) for every \(u\) in \([0,1]\).

\(^2\) For further details, see Embrechts, Lindskog and Mcneil (2003) and Nelsen (2006).
Equivalently, an n-copula (or n-dimensional copula) is a function $C : [0,1]^n \rightarrow [0,1]$ with the following properties:

3. For all $u$ in $[0,1]^n$, $C(u) = 0$ if at least one coordinate of $u$ is 0, $C(u) = u_k$ if all the coordinates of $u$ are 1 except $u_k$.

4. For all $a$ and $b$ in $[0,1]^n$ such that $a_i \leq b_i$ for every $i$, $V_C([a,b]) \geq 0$, where $V_C$ is called $C$-volume.

One of the main results of the theory of copulas is Sklar’s Theorem (1956) which we state below.

**Theorem 1**: (Sklar’s Theorem). Let $X_1,\ldots,X_n$ be random variables with distribution functions $F_1,\ldots,F_n$, respectively, and joint distribution function $H$. Then, there is a $C$ such that

$$H(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n)). \quad (2)$$

for every $x_1,\ldots,x_n \in \mathbb{R}^n$. If $F_1,\ldots,F_n$ are all continuous, then $C$ is unique; otherwise $C$ is determined only on $\text{Im} F_i \times \cdots \times \text{Im} F_n$. Reciprocally, if $C$ is an n-copula and $F_1,\ldots,F_n$ are distribution functions, then the function $H$ defined above is an n-dimensional distribution function with margins $F_1,\ldots,F_n$.

**Corollary 1**: Let $H$ be an n-dimensional distribution function with continuous margins $F_1,\ldots,F_n$ and copula $C$ (where $C$ is defined as in Theorem 1). Thus, for any $u = (u_1,u_2,\ldots,u_n)$ in $[0,1]^n$,

$$C(u_1,\ldots,u_n) = H(F_1(u_1),\ldots,F_n(u_n)) \quad (3),$$

where $F_i^{(-1)}$ is the generalized inverse.

The theorem above allows more flexibility for modeling multivariate distributions. Since a copula is a function that binds a multivariate distribution function to its marginal distributions of any dimension, it possesses all the relevant information about the dependence structure between the random variables. This adds greater flexibility in modeling multivariate distributions and their margins.

In many cases, the dependence pattern and the general dynamics of a time series can be captured through conditional distributions on past observations, as observed by Mendes (2005) and Patton (2006). Thus, the extension of Sklar's theorem for the conditional case can prove to be very useful. According to Patton (2006), the Sklar's theorem for the conditional case is defined as follows:

**Theorem 2**: (Sklar's Theorem – Conditional case). Let $X_i | W$, for $i = 1,\ldots,n$, be random variables with conditional distribution functions $F_i$, respectively, and conditional joint distribution function $H$ of $X | W$.

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3 The proof of Sklar’s Theorem can be found in Sklar (1959), Schweiser and Sklar (1983) and Nelsen (2006).

4 $\mathbb{R} = [-\infty, +\infty]$ is the extended real line.


6 Let $F : \mathbb{R} \rightarrow [0,1]$ be a function of univariate distribution. The generalized inverse (or quasi-inverse) of $F$, $F^{(-1)}$, is given by:

$$F^{(-1)}(t) = \begin{cases} \sup_{x \in \mathbb{R}} \{F(x) = t\}, & t = 0 \\ \inf_{x \in \mathbb{R}} \{F(x) \geq t\}, & 0 < t \leq 1 \end{cases}. $$
where \( X = (X_1, X_2, \ldots, X_n) \) and \( W \) has support \( \Omega \). Then there is a copula \( C \) such that, for any \( \mathbf{x} \in \mathbb{R}^n \) and \( \mathbf{w} \in \Omega \),

\[
H(x_1, \ldots, x_n \mid \mathbf{w}) = C(F_1(x_1 \mid \mathbf{w}), \ldots, F_n(x_n \mid \mathbf{w}) \mid \mathbf{w}).
\] (4)

If \( F_1, \ldots, F_n \) are all continuous then \( C \) is unique, otherwise it is determined only on \( \text{Im}F_1 \times \cdots \times \text{Im}F_n \).

Reciprocally, if \( C \) is an \( n \)-copula and \( F_1, \ldots, F_n \) are distribution functions then, \( H \) as defined above is a conditional \( n \)-dimensional distribution function of with margins \( F_1, \ldots, F_n \).

**Corollary 2:** Let \( H \) be an \( n \)-dimensional conditional distribution function with continuous conditional margins \( F_1, \ldots, F_n \) and copula \( C \) (where \( C \) is defined as in Theorem 2). Thus, for any \( \mathbf{u} \) in \([0,1]^n\) and \( \mathbf{w} \in \Omega \),

\[
C(u_1, \ldots, u_n \mid \mathbf{w}) = H(F_1^{-1}(u_1 \mid \mathbf{w}), \ldots, F_n^{-1}(u_n \mid \mathbf{w}) \mid \mathbf{w}).
\] (5)

Note that the information set \( W \) must be the same for all the margins and the copula. Otherwise, the Definition 1 does not hold and \( C \) is not a "real" copula in the original sense since it violates the third assumption in this definition. In that case, we would have a pseudo-copula as defined by Fermanian and Scaillet (2004). These authors establish Sklar’s theorem version for pseudo-copulas.

The conditional density function associated with the distribution function in (4) can be obtained easily, since \( C \) and \( \mathbf{u} \) are \( n \) differentiable. Considering the two-dimensional case, one can write the conditional density function as follows:

\[
h(x_1, x_2 \mid \mathbf{w}) \equiv \frac{\partial^2 H(x_1, x_2 \mid \mathbf{w})}{\partial x_1 \partial x_2} = \frac{\partial F_1(x_1 \mid \mathbf{w})}{\partial x_1} \cdot \frac{\partial F_2(x_2 \mid \mathbf{w})}{\partial x_2} \cdot \frac{\partial^2 C(F_1(x_1 \mid \mathbf{w}), F_2(x_2 \mid \mathbf{w}) \mid \mathbf{w})}{\partial u_1 \partial u_2},
\] (6)

where \( u_1 = F_1(x_1 \mid \mathbf{w}) \) and \( u_2 = F_2(x_2 \mid \mathbf{w}) \).

The previous result will be used below for deriving maximum likelihood estimators.

### 2.2 Estimation

We can rewrite the density function (6) as follows:

\[
h_t(x_1, x_2 \mid \mathbf{w}; \theta) = f_1(x_1 \mid \mathbf{w}; \theta_1) \cdot f_2(x_2 \mid \mathbf{w}; \theta_2) \cdot c(u_1, u_2 \mid \mathbf{w}; \theta_3),
\] (7)

where \( \theta = [\theta_1, \theta_2, \theta_3] \) is a vector of parameters of the joint density

The functional form (7) suggests that the statistical modeling of the conditional density function \( h \) can be decomposed into two problems: (i) identify the conditional distribution of the margins for

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7 Note that the vector \( \mathbf{w} = (w_1, \ldots, w_n) \) is a realization of the random variable \( W \in \Omega \), which will be defined in Section 3.2.
and (ii) establish a functional form for the copula \( C \). Thus, the log-likelihood function is given by

\[
\sum_{t=1}^{T} \log h_t(x_{1t}, x_{2t} | w; \theta_h) = \sum_{t=1}^{T} \log f_{1t}(x_{1t} | w; \theta_{1t}) + \sum_{t=1}^{T} \log f_{2t}(x_{2t} | w; \theta_{2t}) + \sum_{t=1}^{T} \log c_t(u_{1t}, u_{2t} | w; \theta_c),
\]

or

\[
\ell(\theta) = \ell_{f_1}(\theta_1) + \ell_{f_2}(\theta_2) + \ell_c(\theta_c)
\]

with the maximum likelihood estimator defined as

\[
\hat{\theta}_{MLE} = \max_{\theta \in \Theta} \ell(\theta).
\]

Note that obtaining the maximum likelihood estimator (MLE) can be computationally very intensive, especially in the context we propose here, i.e., assuming the parameter \( \theta_c \) varying over time. However, the estimation of the parameters in (8) can be divided into two stages, using the inference method starting from the margins (IFM) proposed by Joe and Xu (1996). The method consists of estimating the parameters of the margins, \( \hat{\theta}_1 \) e \( \hat{\theta}_2 \), in a first step, and estimating the parameter \( \hat{\theta}_c \), in a second step, using the estimates of \( \hat{\theta}_1 \) e \( \hat{\theta}_2 \). That is,

**Step 1**: estimate \( \hat{\theta}_i = \arg \max \ell_{f_i}(\theta) = \arg \max \sum_{t=1}^{T} \log f_{it}(x_{it} | w; \theta) \), \( i = 1, 2 \).

**Step 2**: estimate \( \hat{\theta}_c = \arg \max \ell(\theta) = \arg \max \sum_{t=1}^{T} \log c_t(F_{1t}(x_{1t} | w; \hat{\theta}_1), F_{2t}(x_{2t} | w; \hat{\theta}_2) | w; \theta_c) \), given Step 1.

Just as the MLE, the estimator by IFM is asymptotically normal and consistent, supporting our choice for this method here.

### 2.3 Tail dependence and parametric copulas

Copulas frequently used in financial literature assume elliptical forms, mostly Gaussian (Normal) and T-student. They are called elliptical because they are associated with a quadratic form of correlation between the margins. The dependence structure associated with this family of copulas is the well known Pearson correlation coefficient, which belongs to the interval \([-1,1]\). As a result, these distributions are symmetric.

Other copulas that have been used are Archimedean, which, depending on the functional form of the generating\(^8\) factor associated with them, may have the dependence measure belonging to the most diverse ranges of variation. An example may be given by the copulas used in the extreme value theory (EVT), where we can cite the Gumbel (1960) copula. It allows only positive dependence structures (or the upper tail), for which the parameter belongs to the interval \([1, +\infty]\). This diversity of copula functions with specific dependence structures would make it impossible to compare different functional forms for

\(^8\) This generating factor for Archimedean copulas needs to meet some criteria like convexity, among others In order to also satisfy the assumptions on copulas seen in section 2. See Nelsen (2006).
copulas. To make this comparison possible we focus on a dependence measure known as Tail Dependence\(^9\) (dependence on the tail)\(^10\), which is defined below.

**Definition 2:**

*If the limit*

\[
\lim_{\varepsilon \to 0} \Pr [U_1 \leq \varepsilon \mid U_2 \leq \varepsilon] = \lim_{\varepsilon \to 0} \Pr [U_2 \leq \varepsilon \mid U_1 \leq \varepsilon] = \lim_{\varepsilon \to 0} C(\varepsilon, \varepsilon) / \varepsilon = \tau^L \text{ exists, the copula } C \text{ has a lower tail dependence if } \tau^L \in (0,1]. \text{ Otherwise, the copula has no lower tail dependence.}
\]

*Similarly, if the limit*

\[
\lim_{\delta \to 1} \Pr [U_1 > \delta \mid U_2 > \delta] = \lim_{\delta \to 1} \Pr [U_2 > \delta \mid U_1 > \delta] = \lim_{\delta \to 1} (1 - 2\delta + C(\delta, \delta)) / (1 - \delta) = \tau^U \text{ exists, the copula } C \text{ has upper tail dependence if } \tau^U \in (0,1]. \text{ Otherwise, the } C \text{ copula has no upper tail dependence.}
\]

We use this measure having in mind the financial stylized facts and also due to the results we have found for the marginal distributions (to be seen in Section 3.2), which show some left asymmetry. This may also suggest an asymmetry to the left in the bivariate distribution.

To capture the dependence time dynamics in the dependence parameter follow Patton (2006), where the parameter in question evolves according to an equation with a functional form defined \textit{a priori}. This functional form will be described later in accordance with the selected copulas.

In this paper, we use four bivariate copulas to analyze the dependence between the indexes: Rotated Gumbel, SJC (Symmetrized Joe-Clayton), Normal and T-Student. Their definitions follow below.

### 2.3.1 Rotated Gumbel Copula (RGC) (or Survival Gumbel Copula), which is the complement ("Probability of survival") of the Gumbel copula.

The RGC has the following form:

\[
C_{RG}(u_1, u_2 \mid \theta) = u_1 + u_2 - 1 + C_G(1 - u_1, 1 - u_2 \mid \theta),
\]

where \(C_G\) corresponds to the Gumbel copula:

\[
C_G(u_1, u_2 \mid \theta) = \exp\left(-\left((-\log u_1) + (-\log u_2)\right)^{1/\theta}\right), \quad \theta \in [1, \infty).
\]

The time dynamics equation for the dependence parameter in RGC, \(\theta_t\), is given by

\[
\theta_t = \Lambda \left( \omega_G + \beta_G \theta_{t-1} + \alpha_G \cdot \frac{1}{10} \sum_{j=1}^{10} |u_{1,j-1} - u_{2,j-1}| \right),
\]

where \(\Lambda(x) = 1 + x^2\) is a polynomial transformation to ensure that \(\theta_t \in [1, \infty)\).

The RGC has only lower tail dependence (\(\tau^U = 0\)), which can be obtained by \(\tau^L = 2 - \frac{1}{\theta}\).

### 2.3.2 Symmetrised Joe-Clayton Copula (SJC) (Patton, 2006): the functional form for the SJC copula is

\(^10\)Other measures of dependence have been used to try to compare the dependence between copulas, as the measures of concordance of Kendall’s \(\tau\), Spearman’s \(\rho\) and the Gini’s co-gradation index. Nelsen (2006).
\[ C_{\text{SJC}}(u_1, u_2 \mid \tau^U, \tau^L) = 0.5 \cdot \left( C_{\text{JC}}(u_1, u_2 \mid \tau^U, \tau^L) + C_{\text{JC}}(1-u_1, 1-u_2 \mid \tau^U, \tau^L) + u_1 + u_2 - 1 \right), \]

where \( C_{\text{JC}} \) is the Joe-Clayton copula, also called “BB7”, given by

\[ C_{\text{JC}}(u_1, u_2 \mid \tau^U, \tau^L) = 1 - \left( 1 - \left( 1 - u_1 \right)^{\kappa} + \left[ 1 - \left( 1 - u_2 \right)^{\kappa} \right]^{\gamma} - 1 \right)^{-1/\kappa}, \]

with

\[ \kappa = 1 / \log_2 (2 - \tau^U), \]
\[ \gamma = -1 / \log_2 (\tau^L), \]
\[ \tau^U, \tau^L \in (0, 1). \]

The SJC has an upper and lower tail dependence parameters. Its own dependence parameters, \( \tau^U \) and \( \tau^L \), are the measures of dependence on the upper and lower tail, respectively. Furthermore, \( \tau^U \) and \( \tau^L \) range freely and are not dependent on each other.

The time dynamics equations for the parameters \( \tau^U \) and \( \tau^L \) are

\[ \tau^U_t = \Lambda \left( \omega_u + \beta_u \tau^U_{t-1} + \alpha_u \cdot \frac{1}{10} \sum_{j=1}^{10} \left| u_{t,j} - u_{2,j} \right| \right) \]

and

\[ \tau^L_t = \Lambda \left( \omega_l + \beta_l \tau^L_{t-1} + \alpha_l \cdot \frac{1}{10} \sum_{j=1}^{10} \left| u_{t,j} - u_{2,j} \right| \right), \]

where \( \Lambda(x) = (1 + e^{-x})^{-1} \) is a logistic transformation to maintain \( \tau^L \) and \( \tau^U \) in the interval \((0, 1)\).

2.3.3 Normal Copula (N): we can define the Normal Copula associated with the bivariate normal as follows:

\[ C_N(u_1, u_2 \mid \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left\{ -\frac{(r^2 - 2\rho rs + s^2)}{2(1 - \rho^2)} \right\} dr ds, \quad \rho \in (-1, 1), \]

where the dependence parameter \( \rho \), is the coefficient of linear correlation. Its dynamic equation is

\[ \rho_t = \Lambda \left( \omega_N + \beta_N \rho_{t-1} + \alpha_N \cdot \frac{1}{10} \sum_{j=1}^{10} \Phi^{-1}(u_{t,j}) \cdot \Phi^{-1}(u_{2,j}) \right), \]

where \( \Lambda(x) = (1 - e^{-x})(1 + e^{-x})^{-1} \) is a logistic transformation to keep \( \rho \in (-1, 1) \). The Normal copula does not have tail dependence, that is \( \tau^L = \tau^U = 0 \).

2.3.4 T-student Copula (T): it is associated with the bivariate t-student and is given by
\[
C_T(u_1, u_2 \mid \rho, \nu) = \int_{-\infty}^{\nu^{-1}(u_1)} \int_{-\infty}^{\nu^{-1}(u_2)} \frac{1}{2\pi \sqrt{1-\rho^2}} \left(1 + \frac{r^2 - 2\rho r s + s^2}{v(1-\rho^2)}\right)^{-\frac{v+2}{2}} dr ds,
\]

where the parameters \( \rho \) e \( \nu \) are the coefficient of linear correlation and the degrees of freedom, respectively. In addition, their time evolution is given by

\[
\rho_t = \Lambda \left( \omega_{\rho} + \beta_{\rho} \rho_{t-1} + \alpha_{\rho} \cdot \frac{1}{10} \sum_{j=1}^{10} T^{-1}_v(u_{i,j-1}) \cdot T^{-1}_v(u_{2,j-1}) \right)
\]

and

\[
\nu_t = \tilde{\Lambda} \left( \omega_{\nu} + \beta_{\nu} \nu_{t-1} + \alpha_{\nu} \cdot \frac{1}{10} \sum_{j=1}^{10} T^{-1}_v(u_{i,j-1}) \cdot T^{-1}_v(u_{2,j-1}) \right),
\]

where \( \Lambda(x) = (1 - e^{-x})(1 + e^{-x})^{-1} \) is again a logistic transformation to constrain \( \rho \in (-1, 1) \) and \( \tilde{\Lambda}(x) = (e^x / (1 + e^x)) \cdot 98 + 2 \) is another transformation to maintain \( \nu \in [2,100] \).

The T-copula has symmetrical tail dependence with \( \tau^T = \tau^\nu = 2T_{v+1} \left( -\frac{(v+1)(1-\rho)}{1+\rho} \right) \), where \( T_{v+1} \) is the t-student cumulative distribution with \( v+1 \) degrees of freedom.

Please note that the above evolution equations follow a kind of restricted ARMA (1,10) process, which is composed of an autoregressive component, to capture any persistence of dependence, and a forcing variable. Following Patton (2006), this forcing variable is the mean absolute difference between \( u_{i,t} \) e \( u_{2,t} \) over the previous 10 observations (to the copulas RGC and SJC) and the mean of the result of \( \Phi^{-1}(u_{i,j-1}) \cdot \Phi^{-1}(u_{2,j-1}) \) and \( T^{-1}_v(u_{i,j-1}) \cdot T^{-1}_v(u_{2,j-1}) \), the transformed variables, for the Normal and T over the previous 10 observations.

The idea, or intuition, behind this forcing variable is to capture the combined movement of the transformed margins processed over 10 past observations. If \( u_{i,t} \) e \( u_{2,t} \) are comonotonic (have a perfect positive association), the distance between them is close to zero\(^{12}\). Then, this mean absolute difference gives an idea of how the data behave in relation to co-monotonicity. In other words, “the expectation of this distance measure is inversely related to the concordance ordering of copulas; under perfect positive dependence it will equal zero, under independence it equals 1/3, and under perfect negative dependence it equals 1/2.” (Patton, 2006).

### 3 Modelling dependence between market indexes: definitions and first empirical results

In this section we start by describing our data set and highlighting its main features. Then, in our first modelling stage, we model and estimate the margins of the four studied indexes considering both their conditional mean and variance. In the next stage, we estimate several copulas.

\(^{11}\) Where \( \Phi^{-1} \) e \( T^{-1}_v \) are the inverse of the Normal and t-student c.d.f.

\(^{12}\) For elliptical copulas, we used the mean of marginal transformed products but, the idea is the same. The purpose of the introduction of these products has been to make the results comparable. See Patton (2006).
3.1 Data description

In our analysis, we use four indexes of financial markets: FTSE100 (Financial Times and London Stock Exchange Index), IBOVESPA (Índice da Bolsa de Valores de São Paulo), IPCMX (Índice de Precios y Cotizaciones de Bolsa Mexicana de Valores) and S&P500 (Standard and Poor 500 Index). The data is composed of daily observations, comprising the period from January 1st, 1999 to February 20th, 2008, with a total of 2103 points. During this period Brazil went on to adopt both the flexible exchange rates and the inflation targeting systems, besides facing other events that disturbed the world markets in general, such as the terrorist attacks on September 11th, 2001, Brazilian presidency elections in the end of 2002 and the real estate crisis in the United States that started around the end of 2006/beginning of 2007.

Table 1 shows some descriptive statistics for the log-returns of such data. We can see that this data usually shows signals of both negative asymmetry and kurtosis excess (except the IBOV – IBOVESPA -, which shows positive asymmetry). Also according to Jarque-Bera test statistics, there is evidence that the log-returns are not normally distributed. Meanwhile, estimated mean and standard deviation are visually very similar. These statistics are in line with what is described in the literature on financial data.

<table>
<thead>
<tr>
<th></th>
<th>FTSE100</th>
<th>IBOV</th>
<th>IPCMX</th>
<th>SP500</th>
</tr>
</thead>
<tbody>
<tr>
<td>N° of Obs.</td>
<td>2103</td>
<td>2103</td>
<td>2103</td>
<td>2103</td>
</tr>
<tr>
<td>Mean</td>
<td>4.25E-06</td>
<td>0.00106</td>
<td>0.00097</td>
<td>4.24E-05</td>
</tr>
<tr>
<td>Median</td>
<td>0.00046</td>
<td>0.00134</td>
<td>0.00137</td>
<td>0.00043</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.05904</td>
<td>0.28832</td>
<td>0.07493</td>
<td>0.05574</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.0606</td>
<td>-0.1223</td>
<td>-0.0966</td>
<td>-0.06</td>
</tr>
<tr>
<td>Std. deviation</td>
<td>0.01197</td>
<td>0.02064</td>
<td>0.0154</td>
<td>0.01176</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.1934</td>
<td>1.00909</td>
<td>-0.1102</td>
<td>-0.0579</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>5.61255</td>
<td>22.1473</td>
<td>5.91558</td>
<td>5.39423</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>611.183</td>
<td>32481.9</td>
<td>749.125</td>
<td>503.47</td>
</tr>
</tbody>
</table>

The estimates of multivariate patterns are performed in pairs. More precisely, the estimation of the joint distribution via copulas is carried out between the IBOV and each of the other indexes in pairs. These results will be discussed later. The main goal at this point is to measure and evaluate the dependence structure between the IBOV and the other indexes. These are then the guidelines for the following discussions.

3.2 Modelling of marginal distributions

We study several models for the margins, both for the conditional mean and for the conditional variance. For this univariate setting we assume normal, t-student and t-skewed errors, with the mean equations following autoregressive and/or moving averages processes, and variance equations following EGARCH, TGARCH and PGARCH processes, among others13. The choice of the most appropriate model for the margins is made taking into consideration the probability integral transform. That is, if the

---

13 The estimates of these models will not be reported here due to physical space limitations, but the authors make them accessible if requested.
marginal is well specified, then the probability integral transform of the residuals will have uniform distribution \([0, 1]\), a result needed to identify conditional copulas\(^{14}\).

Another important result when dealing with copulas is that the information set to which the model is conditioned upon must be the same for each marginal and underlying copula. To test the validity of this assumption, we add lagged log-returns from other variables and conditional residuals from other models\(^{15}\) into the mean and variance equations, that is, we estimate models to the margins as follows:

\[
X_{it} = ARMA(1,1) + \gamma_{x_{i1}} X_{j_{i-1}} + \gamma_{x_{i2}} \epsilon_{j_{i-1}} + \epsilon_{it} \\
\sigma_{x_{i}}^{2} = \omega_{x_{i}} + \beta_{x_{i}} \sigma_{x_{j_{i-1}}}^{2} + \alpha_{x_{i}} \epsilon_{j_{i-1}}^{2} + \gamma_{x_{i2}} \epsilon_{j_{i-1}}^{2}, \quad j \neq i,
\]

where \(\epsilon_{it}\) and \(\epsilon_{i_{i-1}}\) are the residuals and one lagged residuals of the model \(i\), \(\epsilon_{j_{i}}\) and \(\epsilon_{j_{i-1}}\) are the residuals and one lagged residuals of the model \(j\). Note that the vector of information to which the margins have been conditioned upon is \(w = (X_{i_{i-1}}, X_{2_{i-1}}, \epsilon_{i_{i-1}}, \epsilon_{2_{i-1}})\).

The results of these estimates are such that all coefficients \(\gamma_{x_{j}}\)'s are not statistically significant. That is, according to our analysis, the log-return of IBOV is not affected conditionally by the sets of information from the other indexes. Therefore, this suggests that the information passed on from the behavior of other indexes does not affect neither the conditional mean nor the conditional variance of IBOV (given its own information set) and may be dropped from the marginal modeling and, consequently, from the estimates of underlying copulas\(^{16}\).

Thus, the indexes log-returns could be modeled, individually, as ARMA (1,1)-GARCH (1,1) processes with skewed-t errors\(^{17}\) \((v_{i}, \lambda_{i})\). That is,

\[
X_{it} = ARMA(1,1) + \epsilon_{it} \\
\sigma_{x_{i}}^{2} = \omega_{x_{i}} + \beta_{x_{i}} \sigma_{x_{j_{i-1}}}^{2} + \alpha_{x_{i}} \epsilon_{j_{i-1}}^{2} + \gamma_{x_{i2}} \epsilon_{j_{i-1}}^{2}, \\
\epsilon_{it} \sim \text{skewed}-t(v_{i}, \lambda_{i})
\]

where the skewed-t density is given by

\[
g(z | v, \lambda) = \begin{cases} 
bc \left(1 + \frac{1}{v-2} \left(\frac{hz + a}{1-\lambda}\right)^{2}\right)^{-\left(v+1\right)/2} & z < -a/b \\
bc \left(1 + \frac{1}{v-2} \left(\frac{hz + a}{1+\lambda}\right)^{2}\right)^{-\left(v+1\right)/2} & z \geq -a/b
\end{cases}, \quad (11)
\]

with the constants \(a\), \(b\) and \(c\) defined as

\[
a = 4\lambda c \left(\frac{v-2}{v-1}\right), \quad b^{2} = 1 + 3\lambda^{2} - a^{2}, \quad c = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi(v-2)\Gamma\left(\frac{v}{2}\right)}},
\]

\(^{14}\) As seen in Section 2.

\(^{15}\) We estimated the bivariate distribution of the pairs.

\(^{16}\) See Patton (2006).

\(^{17}\) The Skewed-t GARCH model was defined by Hansen (1994).
where the parameters $\nu$ and $\lambda$ representing the degrees of freedom and asymmetry, respectively.

Table 2 shows the results of the margins fit. We can see that all margins seem to be asymmetric because for all sets the parameter $\lambda_i$ is significantly different of 0. The equation specified for the variance is also significant (except for the constant $\omega_i$, which is not significant - nevertheless we choose not to remove it from the model, due to its long-term interpretation and to that the properties of least squares estimation can be hold). Moreover, there is no autocorrelation in the residuals, neither in the square of the residuals, as shown by $Q$ of Lijung-Box statistics, which indicates that the model is well specified.

In addition, the Kolmogorov-Smirnov tests, used to test the marginal probability transform indicates no evidence that they are not uniform $[0,1]$. This result allows us to consider that the marginal distributions are adequately specified, a necessary condition to our estimation.

### Table 2: Results for the marginal distributions

<table>
<thead>
<tr>
<th>$\hat{\theta}_i \setminus i$</th>
<th>IBOV</th>
<th>SP500</th>
<th>FTSE100</th>
<th>IPCMX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.000008</td>
<td>0.001013**</td>
<td>-0.000006</td>
<td>0.000971*</td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.679380***</td>
<td>0.029651</td>
<td>0.645140***</td>
<td>0.006686</td>
</tr>
<tr>
<td>MA(1)</td>
<td>-0.728150***</td>
<td>0.006085</td>
<td>-0.714080***</td>
<td>0.076289</td>
</tr>
<tr>
<td>$\omega_i$</td>
<td>0.000018</td>
<td>0.000002</td>
<td>0.000002</td>
<td>0.000008</td>
</tr>
<tr>
<td>$\sigma_{x,t-1}^2$</td>
<td>0.889050***</td>
<td>0.884700***</td>
<td>0.846700***</td>
<td>0.853870***</td>
</tr>
<tr>
<td>$\xi_{it-1}^2$</td>
<td>0.059046***</td>
<td>0.105970*</td>
<td>0.138190**</td>
<td>0.114880***</td>
</tr>
<tr>
<td>$\nu_i$</td>
<td>10.913000**</td>
<td>8.253000</td>
<td>25.518000</td>
<td>6.521100**</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>-0.094223***</td>
<td>-0.121500**</td>
<td>-0.186970**</td>
<td>-0.058466**</td>
</tr>
<tr>
<td>$Q(20)$</td>
<td>0.2085</td>
<td>0.1616</td>
<td>0.1697</td>
<td>0.3687</td>
</tr>
<tr>
<td>$Q^2(20)$</td>
<td>0.9536</td>
<td>0.7299</td>
<td>0.9999</td>
<td>0.5721</td>
</tr>
<tr>
<td>K-S Test</td>
<td>0.7895</td>
<td>0.4845</td>
<td>0.8901</td>
<td>0.9040</td>
</tr>
</tbody>
</table>

Note: (***) (**) (*) – significant to 1%, 5% and 10% respectively.

$Q$ – $p$-value of the Lijung-Box test ($H_0$: null autocorrelation up to the twentieth lag).

K-S – $p$-value of the Kolmogorov-Smirnov test ($H_0$: uniform($0,1$) distribution).

### 3.3 Copula Modelling

With the results from the marginal distributions fit, we start choosing the most appropriate copula function to express the bivariate distribution between the IBOV and each of the other indexes. The focus of our analysis via copulas is to observe the behavior over time of the dependence parameter for each copula function used. However, this dependence parameter is different for each function. That is, each

---

18 All estimates in this study were made using the toolboxes statistics e optimization (Matlab 7®), UCSD-GARCH toolbox provided by Kevin Shepard (University of Oxford – Department of Economics), the copula toolbox provided by Andrew Patton (University of Oxford – Department of Economics) and some functions created by the authors for the Matlab 7® software.
copula function captures specific properties of this structure associated with the margins. We here use those copulas described in Section 2.3.

Table 3 shows the information criteria used to initially select the copulas above (regarding the dependence structure between IBOV and the other indexes). They are the Bayesian Information (BIC), Akaike Information (AIC) and the negative log-likelihood (LL). According to these criteria, we observe that the SJC copula is the function that best fits the pairs IBOV/SP500 and IBOV/IPCMX, while the T-copula fits better the pair IBOV/FTSE100. Note that, for a specific pair of indexes, all criteria select (smaller value) the same copula as having the best fit. This was possibly because the sample is relatively large (2103 records). Among the evaluated copulas, the Normal is one with the "worst" fit.

Table 3: Information criteria for the estimated copulas

<table>
<thead>
<tr>
<th></th>
<th>AIC</th>
<th>BIC</th>
<th>LL</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBOV/IPCMX</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SJC</td>
<td>-859.37</td>
<td>-859.35</td>
<td>-429.69</td>
</tr>
<tr>
<td>T-student (Constant)</td>
<td>-815.92</td>
<td>-815.91</td>
<td>-407.96</td>
</tr>
<tr>
<td>T-student</td>
<td>-814.86</td>
<td>-814.84</td>
<td>-407.43</td>
</tr>
<tr>
<td>RGC</td>
<td>-806.02</td>
<td>-806.01</td>
<td>-403.01</td>
</tr>
<tr>
<td>Normal</td>
<td>-798.46</td>
<td>-798.45</td>
<td>-399.23</td>
</tr>
<tr>
<td>IBOV/SP500</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SJC</td>
<td>-900.06</td>
<td>-900.04</td>
<td>-450.03</td>
</tr>
<tr>
<td>RGC</td>
<td>-871.55</td>
<td>-871.54</td>
<td>-435.78</td>
</tr>
<tr>
<td>T-student</td>
<td>-858.29</td>
<td>-858.28</td>
<td>-429.15</td>
</tr>
<tr>
<td>T-student (Constant)</td>
<td>-848.84</td>
<td>-848.83</td>
<td>-424.42</td>
</tr>
<tr>
<td>Normal</td>
<td>-839.55</td>
<td>-839.54</td>
<td>-419.78</td>
</tr>
<tr>
<td>IBOV/FTSE100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T-student</td>
<td>-327.10</td>
<td>-327.09</td>
<td>-163.55</td>
</tr>
<tr>
<td>SJC</td>
<td>-325.66</td>
<td>-325.65</td>
<td>-162.83</td>
</tr>
<tr>
<td>RGC</td>
<td>-320.53</td>
<td>-320.52</td>
<td>-160.27</td>
</tr>
<tr>
<td>T-student (Constant)</td>
<td>-315.53</td>
<td>-315.53</td>
<td>-157.77</td>
</tr>
<tr>
<td>Normal</td>
<td>-310.90</td>
<td>-310.89</td>
<td>-155.45</td>
</tr>
</tbody>
</table>

Note: AIC - Akaike Information Criteria; BIC - Bayesian Information Criteria; LL - copula log-likelihood. Highlighted, the selected model.

Table 4 presents the results of the parameter estimates for the time-varying copula. From them we can make the following observations about the dependence structure:

1. Normal copula - for the pair IBOV/FTSE100 - all estimates are significant. For the other pairs, only the estimation $\beta_N$ is significant. This suggests a first-order autocorrelation for the dependence parameters;

---

19 A goodness-of-fit test criterion will be discussed in details in the next section.
20 Besides these four time-variants copula, we estimated nine others copula: the four static copula corresponding to the time-variant used and the Frank, Clayton, Rotated Clayton, Plackett and Gumbel copulas, too static. However, the time-varying copulas have the best results according to information criteria AIC, BIC and LL. As the proposal's work is to discuss the dependence over time did not report these results (available upon request).
21 Note that the sign of the value of the maximum likelihood is negative. This happened because in our computational routines we minimize the negative of likelihood, just for convenience.
2. T and RGC copulas - the significance of the estimates provides evidence that these copulas match well with the pairs in question (only the estimates to $\beta_G$ in the relation IBOV/IPCMX, $\beta_T$ to IBOV/SP500 and $\alpha_T$ to IBOV/FTSE100 are not significant);

3. SJC copula - the significance of the estimates seems to confirm the selection made by the information criteria AIC, BIC and LL. This copula fits well the pairs IBOV/IPCMX and IBOV/SP500;

4. For the pair IBOV/SP500, considering the copula selected by the information criteria, the SJC copula, the only non-significant parameter is $\beta_L$. This suggest lack of autocorrelation in the lower tail;

5. All estimated coefficients for the forcing variable, which tries to capture the combined movement of the marginal probability transforms, are significant for the copula indicated by the information criteria (except for the T-copula in the pair IBOV/FTSE100). This may indicate that this combined behavior in earlier periods is important for the dependence analysis;

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>IBOV/IPCMX</th>
<th>IBOV/SP500</th>
<th>IBOV/FTSE100</th>
<th>IBOV/IPCMX</th>
<th>IBOV/SP500</th>
<th>IBOV/FTSE100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_N$</td>
<td>0.32985</td>
<td>0.29049</td>
<td>1.1404***</td>
<td>$\omega_G$</td>
<td>1.2547*</td>
<td>0.22174***</td>
</tr>
<tr>
<td></td>
<td>(0.180793)</td>
<td>(0.456175)</td>
<td>(0.000000)</td>
<td></td>
<td>(0.094347)</td>
<td>(0.000000)</td>
</tr>
<tr>
<td>$\beta_N$</td>
<td>0.30842***</td>
<td>0.14164*</td>
<td>0.46722***</td>
<td>$\beta_G$</td>
<td>-0.088719</td>
<td>0.42721***</td>
</tr>
<tr>
<td></td>
<td>(0.000001)</td>
<td>(0.059382)</td>
<td>(0.000000)</td>
<td></td>
<td>(0.647980)</td>
<td>(0.000000)</td>
</tr>
<tr>
<td>$\alpha_N$</td>
<td>1.3809</td>
<td>1.6408</td>
<td>-1.4715***</td>
<td>$\alpha_G$</td>
<td>-1.6997*</td>
<td>-0.67107***</td>
</tr>
<tr>
<td></td>
<td>(0.216226)</td>
<td>(0.432687)</td>
<td>(0.008836)</td>
<td></td>
<td>(0.054397)</td>
<td>(0.000000)</td>
</tr>
<tr>
<td>$\omega_U$</td>
<td>1.1837***</td>
<td>-1.7856***</td>
<td>0.81694</td>
<td>$\omega_T$</td>
<td>0.088213***</td>
<td>-0.221410***</td>
</tr>
<tr>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
<td>(0.202516)</td>
<td></td>
<td>(0.0035)</td>
<td>(0.000000)</td>
</tr>
<tr>
<td>$\beta_U$</td>
<td>-9.5459***</td>
<td>-1.218***</td>
<td>-9.8472</td>
<td>$\beta_T$</td>
<td>0.12931***</td>
<td>0.04212***</td>
</tr>
<tr>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
<td>(0.287845)</td>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
</tr>
<tr>
<td>$\alpha_U$</td>
<td>-0.23711***</td>
<td>3.9466***</td>
<td>-3.254200</td>
<td>$\alpha_T$</td>
<td>1.9503***</td>
<td>2.6157***</td>
</tr>
<tr>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
<td>(0.190973)</td>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
</tr>
<tr>
<td>$\omega_L$</td>
<td>-1.96***</td>
<td>-1.7261***</td>
<td>2.1506***</td>
<td>$\omega_{2T}$</td>
<td>5.0***</td>
<td>-1.9957***</td>
</tr>
<tr>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
<td>(0.000000)</td>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
</tr>
<tr>
<td>$\beta_L$</td>
<td>-0.36862***</td>
<td>-0.76518</td>
<td>-9.1768</td>
<td>$\beta_{2T}$</td>
<td>-0.79866***</td>
<td>0.23787</td>
</tr>
<tr>
<td></td>
<td>(0.000000)</td>
<td>(0.22684)</td>
<td>(0.161564)</td>
<td></td>
<td>(0.000000)</td>
<td>(0.0323869)</td>
</tr>
<tr>
<td>$\alpha_L$</td>
<td>4.0532***</td>
<td>3.6656***</td>
<td>-5.0038***</td>
<td>$\alpha_{2T}$</td>
<td>-0.070300***</td>
<td>-0.047506***</td>
</tr>
<tr>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
<td>(0.000000)</td>
<td></td>
<td>(0.000000)</td>
<td>(0.000000)</td>
</tr>
</tbody>
</table>

Note: (***) (**), (*) – significant to 1%, 5% e 10% respectively. Between brackets $p$-value ($H_0$: Parameter = 0).

These results allow us to conclude that the joint distributions of the studied pairs must be asymmetric. Also that the functional forms of the employed copulas capture tail dependence.
4 Goodness-of-fit test

Despite AIC, BIC and LL information criteria results in Table 3 indicate which copula could be a good choice, their values are very close among the copulas. Beside this, these information criteria reveals nothing about how good is the model fits to the data. Thus, the problem of deciding which of these copulas is the best “guess” for the data generating process remains. In other words, it is necessary to conduct a more detailed assessment to make our final choice. Some tests of goodness-of-fit have been implemented in the literature to allow the model choice based on copula functions. In this context, some test of goodness-of-fit have been implemented in the literature.

In line with this, here we use a simulation-based method via parametric bootstrap. The test we employ follows the methodology proposed by allcroft and Glasbey (2003), which is used in the context of copulas by Nikoloulopoulos and Karlis (2008).

The approach is designed to simulate data from the candidate models, re-estimate the models using the simulated data and compute the log-likelihood estimates, comparing them with the original log-likelihood estimates. The models with good fits are those whose log-likelihoods in the simulated data are close to the original log-likelihoods. In other words, we follow the following steps:

1. Estimation of parameters for the candidate models using observed data and computation of the maximum likelihood for each model;
2. Simulation of a set of samples using each model in the previous step, re-estimation of the models using these samples and computation of the log-likelihood;
3. Comparison of the log-likelihood of the simulated data with the original log-likelihood.

As stated by Nikoloulopoulos and Karlis (2008), the joint distribution of the log-likelihood estimators is approximately Normal, based on the central limit theorem. Thus, the square Mahalanobis distance can be used to compare the models. This distance is given by

\[ D_k^2 = (\lambda_k - \tilde{\lambda}_k)S^{-1}(\lambda_k - \tilde{\lambda}_k)', \quad k = 1, \ldots, m, \]

where \( \lambda \) is the vector of original maximum log-likelihood estimates from de observed data, \( \tilde{\lambda}_k \) is the vector maximum log-likelihood estimate averages from the simulated data to the model \( k \) and \( S \) is the sample covariance matrix. Assuming normality for these likelihoods, \( mD_k^2 \) has distribution \( F_{m,B} \) (distribution \( F \) with \( m \) and \( B \)-1 degrees of freedom where \( B \) is the number of simulated samples) under the null hypothesis of which the \( k \)-th model is correct (Mardia et al, 1979).

Table 5 shows the averaged p-value for the Mahalanobis distance over 100 simulated samples of size 2103 (size of the observed data). We can see that the T-student copula with constant parameters and Normal time-variant copula have a good fit to the data (the null hypothesis, which is discussed above, is not rejected) in all pairs analyzed for each different source of simulated data. The T-student time-variant copula shows a good fit for all pairs except to the pair IBOV/SP500.

In IBOV/IPCMX, the copula T-student, RGC and SJC time-variant show a good fit when the data are from the SJC copulas and RGC, that is, when the original copula displays tail dependence (higher and lower asymmetry). When the origin is a copula without tail dependence (Normal) or with symmetric tail dependence (T-student), the T-student, RGC and SJC copulas do not have a good adherence to the data (p-value <0.10, the null hypothesis, which the copula is correct, is rejected).

For the pair IBOV/SP500, the copula RGC is chosen only when the data comes from the RGC itself. The SJC is indicated when the origin is a copula that has tail dependence (higher and lower), in this

---

22 See Fermanian (2005) and Dobrié and Schmidt (2007).
case, the SJC and RGC. When the origin is an elliptical copula, only the constant T-student, T-student time-variant and Normal time-variant are chosen.

Observing the results for the pair IBOV/FTSE100, we notice that the T-student time-variant does not have a good fit whatever the origin of the data is. The SJC copula is chosen when the data origin is different from Normal copula, that is, when the data comes from T-student, RGC and SJC. The RGC copula is chosen only when the data origin is the SJC copula or the RGC copula.

We conclude that, even in an asymmetric dependence and time-variant context, T-student constant copula is a good choice when it comes to fitting the data. Similarly, the Normal time-variant copula also offers a good fit to the data. More extreme copulas, as RGC and SJC copulas, which have lower and upper tail dependence, offer a good fit only when the data shows some asymmetric dependence.

Table 5: P-value for the square distance of Mahalanobis between original data and simulated - average over 100 replications

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<thead>
<tr>
<th></th>
<th>Simulated</th>
<th>Original</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Normal</td>
<td>T-student</td>
<td>SJC</td>
</tr>
<tr>
<td>T-student (constant)</td>
<td>0.23896</td>
<td>0.49149</td>
<td>0.99933</td>
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<td>Normal</td>
<td>0.76012</td>
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<td>0.83687</td>
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<tr>
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<td>0.99684</td>
</tr>
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<td>0.00000</td>
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<td>0.99112</td>
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<thead>
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<tbody>
<tr>
<td></td>
<td>Normal</td>
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<td>SJC</td>
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<tr>
<td>T-student (constant)</td>
<td>0.99998</td>
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<th>Original</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Normal</td>
<td>T-student</td>
<td>SJC</td>
</tr>
<tr>
<td>T-student (constant)</td>
<td>0.46333</td>
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<td>Normal</td>
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<td>RGC</td>
<td>0.00000</td>
<td>0.08059</td>
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<tr>
<td>SJC</td>
<td>0.00023</td>
<td>0.97065</td>
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<tr>
<td>T-student</td>
<td>0.00000</td>
<td>0.01518</td>
<td>0.00106</td>
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5 Dependence analysis

Figure 1 shows the time dynamics of the dependence parameters of the four time-varying copulas on the pair IBOV / IPCMX. In panel (a) we have the tail dependence dynamics for the T-copula with
symmetric tail dependence (lower and upper). The dependence coefficient for the T-copula (constant) was 0.1613. The evolution of this parameter over time is rather erratic. Some periods are worth to be highlighted, as for instance between the first semester of 1999 and the second semester of 2001, and also from mid 2006 to early 2008, where the dependence seems more unstable.

In panel (b) we have the dependence parameter for the RGC copula (which shows only lower tail dependence), whose constant coefficient is 0.4425. The evolution of this parameter over time is around this constant most of the observed sample. However, this behavior changes from mid-2006, when the dependence starts rising above 0.4425.

![Figure 1: Tail dependence- IBOV/IPCMX](image)

Note: For the Normal copula the linear correlation coefficient is presented. In red, the dependence parameter for the respective static copula is shown.

The SJC copula, as said previously, allows capturing the tail dependence in both the lower tail and the upper tail. In Panel (c) we have the Lower tail dependence. The coefficient of lower tail dependence (constant SJC) is 0.3952. Between January 1999 and mid-2000, that rate fluctuates around this value. Since then, the evolution of this parameter remains below this constant until January 2006, when it rises and remains around 0.6. Note that this is informative, showing a lower volatility relative to the others, and reveals certain standard behavior for the lower dependence dynamics.

The coefficient of higher tail dependence for SJC copula is shown in Panel (d). The value for the constant is 0.3140, lower than the 0.3952 constant of the lower tail dependence. This result is confirmed by the evolution over time. This was already expected due to the typical behavior of higher dependence on lower tails in financial markets. However, the dependence on the higher tail seems to be very volatile and not very informative.

In the Panel (e) we have the evolution of the linear correlation coefficient for Normal copula. The value for the constant is 0.5529 and time evolution takes place around this constant, overcoming it only from 2007 onwards.

Figure 2 shows the estimated dependence structures for the case IBOV/SP500. We may notice in all those plots that all estimated copulas (except the Normal) indicate an existence of tail dependence.
between IBOV and SP500. The T-copula (constant) registers a coefficient of 0.0923 and its evolution along time, shown on Panel (a), is around this constant coefficient, which indicates that this copula captures relatively low tail dependence time dynamics.

Figure 2: Tail dependence- IBOV/SP500

The RGC copula in Panel (b), which measures the dependence in the lower tail, captures a constant coefficient of 0.4487. The time evolution of the parameter fluctuates around that constant except from mid 2006 onwards. From that moment on, the dependence becomes greater than that recorded previously, and above the constant value.

This behavior is also captured by another copula. The SJC copula (Panels c and d) reveals constant coefficients of 0.3974 and 0.3331 in the lower and higher tail, respectively, and the time evolution shows an increase with the parameters overcoming those constants from mid 2006. Nevertheless, this copula offers additional information. The dependence in the higher tail becomes greater than the dependence in the lower tail from the second half of 2006. This result is reverse to that expected, since an increased dependence in the lower tail is more commonly found than in higher tail. The linear correlation coefficient has a similar behavior, it is around the constant coefficient, 0.5716, and from 2006 the correlation starts to overcome this constant (But at a lower intensity than for the other copulas).

The dependence structure between IBOV/FTSE100 seems more volatile that in the earlier pair. As we can see in Figure 3, the T-copula estimates a constant dependence of 0.02382, while the evolution of the parameter of the varying T-copula (Panel a) ranges from 0 to 0.17. In other words, we can infer that there is a lower dependence time dynamic.

The linear correlation coefficient estimated by the Normal copula (Panel e) evolves around the constant 0.3656. This correlation is lower than those shown for IBOV / IPCMX and IBOV/SP500. This lower association between IBOV/FTSE100 is also captured by another copula. The dependence in the lower tail given by the RGC copula (Panel b) remains constant around its corresponding 0.2889. And the
lower and higher dependence dynamics given by the SJC copula (Panels c and d) remain around their constant coefficients 0.2446 and 0.0999, respectively.

All these dependence parameters are lower than those found for the pairs IBOV/IPCXM and IBOV/SP500. This indicates that the dependence between IBOV and FTSE100 indexes is lower than both the dependence between the IBOV and IPCMX, and the dependence between IBOV and SP500.

All these dependence parameters are lower than those found for the pairs IBOV/IPCXM and IBOV/SP500. This indicates that the dependence between IBOV and FTSE100 indexes is lower than both the dependence between the IBOV and IPCMX, and the dependence between IBOV and SP500.

Figure 3: Tail dependence - IBOV/FTSE100

Note: For the Normal copula is shown the linear correlation coefficient.

Considering the dependence structure that is picked up for these three evaluated pairs, we notice that the degrees of association between the IBOV and the other indexes in question seem to have increased from 2006, suggesting a greater interconnection among these markets.

In addition to observing how the time evolution of the dependence structure in each copula processes, we also evaluate the performance of each copula regarding to the calculation of VaR. Considerations on the VaR from copulas will be presented in the next section.

6 VaR Calculation and evaluation through copulas

To calculate the Value at Risk we construct portfolios with the three pairs of indexes previously analyzed. The composition of the portfolio is chosen arbitrarily as $X_{P,t} = 0.5X_{1,t} + 0.5X_{2,t}$, which means saying that the portfolio is composed by 50% of each index. Other portfolios choices should not interfere on our analysis.

VaR is defined in the following manner: $VaR(\alpha) = \inf \left\{ s : F_{P,t}(s) \geq \alpha \right\}$, that is, it is the largest loss associated with the portfolio, taking into account a level of significance $\alpha$ at a time horizon of $t$ (in
our case, 1 day). To calculate the VaR we need to know the portfolio distribution $F_p$ and choose the quantile associated to $\alpha$. Although $F_p$ can be calculated theoretically from the marginal conditional distributions, it is complicated to reach a closed analytical form for the joint distribution. For this reason we choose to estimate the $\alpha$ quantiles of $F_p$ via simulation.

Hence, we follow the procedure described below:

- Simulate the joint data via the subjacent copula for each fixed $t$ and obtain estimates for the $\text{VaR(}\alpha\text{)}$;
- Calculate the proportion of the values of $X_p$, which in the sample are smaller than $VaR(\alpha)$, If the marginal distributions and the copula were well specified, this proportion should be close to $\alpha$.

To simulate random vectors from copulas, we used the conditional sampling\textsuperscript{23} method as follows. 

Let the conditional distribution of $U_2$ given $U_1$ be

$$c_m(u_2) = P(U_2 \leq u_2 \mid U_1 = u_1) = F_{U_2 \mid U_1}(u_2 \mid u_1) = \lim_{\Delta u_i \to 0} \frac{C(u_1 + \Delta u_i, u_2) - C(u_1, u_2)}{\Delta u_i} = \frac{\partial}{\partial u_1} C(u_1, u_2) = C_{u_1}(u_2)$$

where $C_{u_1}(u_2)$ is the partial derivative of the copula function in relation to $u_1$. It allows us to generate pairs $(u_1, u_2)$ in the following manner:

1. Generate two independent Uniform$(0.1)$ random variables, $u_1$ and $t$;
2. Consider $u_2 = c_m^{(-1)}(t)$, where $c_m^{(-1)}$ is the inverse generalized of $c_m$;
3. Note that as $c_m(t) = F_{U_2 \mid U_1}(t \mid u_1)$ and $t$ is an observation of a standard uniform random variable, $u_2 = c_m^{(-1)}(t)$ is an observation of the random variable $U_2 \mid U_1 = u_1$. Hence we have that the pair $(u_1, u_2)$ is an observation of the random vector $(U_1, U_2)$, which has the joint distribution $C$;
4. Transform the uniform random vector $(u_1, u_2)$ into the desired vector $(x_1, x_2)$, using the inverse $x_1 = F_1^{(-1)}(u_1) \text{ e } x_2 = F_2^{(-1)}(u_2)$ where $F_1$ and $F_2$ are the estimated marginal distributions as in Section 3.2.

Observe that the marginal distributions $F_1$ and $F_2$ are defined as an ARMA$(1,1)$ plus a skewed t-GARCH$(1,1)$. Thus, to obtain the log-returns after simulating the vector $(u_1, u_2)$, we use

$$(x_1, x_2) = \left(\hat{\mu}_{t,1} + y_{r,1} \cdot \sqrt{\hat{\sigma}_{s,1}^2} + \hat{\mu}_{t,2} + y_{r,2} \cdot \sqrt{\hat{\sigma}_{s,2}^2}\right),$$

where $y_{r,1}$ is the skewed $t(\nu, \lambda)$ inverse of $(u_1, u_2)$ and

$$(\hat{\mu}_{t,1}, \hat{\sigma}_{s,1}, \nu, \lambda)$$

are estimated according to (10).

To test the VaR estimates we used the Kupiec (1995) and the Christoffersen (1998) tests. These tests have the same idea. If the probability of exceeding the VaR is $P^* = Pr(X_p < VaR(\alpha))$, the test is conducted under the null hypothesis $P^* = \alpha$ against the alternative hypothesis $P^* \neq \alpha$.

Kupiec test (1995), also called unconditional coverage, is a likelihood ratio test defined as

\textsuperscript{23} For more details about generation of random vectors via copulas, see Nelsen (2006) and Cherubini, Luciano and Vecchiato (2004).
\[ \text{LR}_k = -2 \cdot \ln[(1 - \alpha)^{T-N} \alpha^N] + 2 \cdot \ln[(1 - N / T)^{T-N} (N / T)^N] \]

\[ \text{LR}_k \sim \chi^2(1) \]

where \( N \) is the number of times in which the VaR is exceeded, \( T \) is the size of the sample, \( \alpha \) is the level of significance and \( N/T \) is the estimated proportion of excesses which we call \( \hat{p}_\alpha = N / T \).

The conditional coverage test by Christoffersen (1998) is also a likelihood ratio test which is given by:

\[ \text{LR}_c = -2\ln[(1 - \alpha)^{T-N} \alpha^N] + 2\ln[(1 - \pi_{01})^{\pi_{01}} \pi_{01}^0 (1 - \pi_{11})^{\pi_{11}} \pi_{11}^0] \]

\[ \pi_{ij} = \sum_j n_{ij} \]

\[ \text{LR}_c \sim \chi^2(2) \]

where \( n_{ij} \) is the number of observations with \( i \) value followed by \( j \), with \( i, j = 0,1 \) (0 means no VaR violation and 1 VaR violation\(^{24} \)), and \( \pi_{ij} \) are the corresponding estimated probabilities. The test considers the estimated proportion of excesses \( \hat{p}_\alpha = \pi_{01} = \pi_{11} \), which means that, if the \( N \) losses are independent, the probability of observing a new violation of VaR if has occurred a violation, or not, in the earlier instant must be the same.

In our experiment for the VaR computation we simulate 1000 replications for each \( t \) and for each of the three portfolios. Table 6 shows the results of Christoffersen (1998) and Kupiec (1995) tests for the VaR calculated on the basis of our simulations. At first, all copulas can be considered to lead to good approximations of the joint distribution in estimating the VaR, as shown by the CT and KT tests (the null hypothesis could not be rejected). This result is consistent with others in the literature. When the margins are well specified, the underlying copula has a minor role in the context of VaR estimation. As noted by Marshall and Zeevi (2002), Chen, Fan, and Patton (2004) and Fantazzini (2006), it is sufficient to obtain good VaR estimates with a constant Normal copula.

Table 6: VaR violation percentage \((N/T)\), Christoffersen (CT) and Kupiec (KT) Tests

<table>
<thead>
<tr>
<th>Copula</th>
<th>( \alpha )</th>
<th>IBOV/IPCXM</th>
<th>IBOV/SP500</th>
<th>IBOV/FTSE100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( N/T )</td>
<td>CT</td>
<td>KT</td>
<td>( N/T )</td>
</tr>
<tr>
<td>SJC copula</td>
<td>1%</td>
<td>0.8563%</td>
<td>0.31917</td>
<td>0.4973</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>5.0428%</td>
<td>0.840436</td>
<td>0.92833</td>
</tr>
<tr>
<td>RJC copula</td>
<td>1%</td>
<td>0.7136%</td>
<td>0.657784</td>
<td>0.16425</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>5.138%</td>
<td>0.911038</td>
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<td>T copula</td>
<td>1%</td>
<td>1.0942%</td>
<td>0.293145</td>
<td>0.66898</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>5.0428%</td>
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<tr>
<td>Normal copula</td>
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<td>0.293145</td>
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<tr>
<td></td>
<td>5%</td>
<td>5.0904%</td>
<td>0.67137</td>
<td>0.84962</td>
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</table>

Note: The values shown for the tests are their respective \( p \)-value.

The only significant result (setting the level of significance of the test at 10\%) according to the CT test is the estimated VaR for RGC copula, in which case, considering a VaR at 5\%, the null hypothesis is rejected.

\(^{24}\) A VaR violation occurs if \( X_{ij} < \text{VaR}(\alpha) \).
Another interesting result is that taking into account the VaR estimates at 1%, the SJC and RGC copula estimates seem more aggressive\textsuperscript{25} than the estimates for the Normal and T copulas (more conservative). At 5%, estimates are conservative for all copulas, apart from the IBOV/SP500 portfolio, where the estimates are aggressive at both 1% and 5%.

**Final remarks**

In this paper we estimate time-varying copula functions to capture the dependence structure and its time dynamics between the IBOVESPA index and each of the following: IPCMX, SP500 and FTSE100 indexes. We conduct our analysis in such a way to observe this bivariate dependence structure in two steps.

Firstly, we fit a univariate ARMA plus skewed-t GARCH models for the margins and test their residuals using the Kolmogorov-Smirnov test. We consider the fit to be good if we accept that the probability integral transform for these residuals are Uniform (0, 1) random variables. For the four indexes under study, according to KS test, we find that these margins are well adjusted and also asymmetric.

In a second step, using estimates of the margins, we fit four different time-varying copula functions, Rotated Gumbel, SJC (Symetrized Joe-Clayton), Normal and T-Student, to estimate the joint distribution of the indexes pairs and observe the behavior of their dependence structures. We point out that there is evidence of an asymmetric dependence between IBOVESPA and the other indexes. Considering the information criteria AIC, BIC and LL, the SJC copula is indicated as the most appropriate function for the pairs IBOV/IPCMX and IBOV/SP500. This suggests the existence of different tail dependences. For the pair IBOV/FTSE100, the copula indicated by the information criteria is the T-student. The Normal copula shows the worst result in all pairs.

Another noticeable result from the time dynamics of the dependence parameter estimates is that the dependence is correlated with the earlier co-movements. That is, as the estimates for the forcing variable are significant for the copula indicated by the information criteria, knowing the joint movement of all the lagged margins in prior periods may be important to make inferences about the dependence at time $t$.

Although they provide an indication of choice among models, the information criteria AIC, BIC and LL do not enable us to conclude which copula gives us the best fit to the data. Moreover, the values of those criteria are close to each other. Accordingly, we use a test of goodness-of-fit based on a parametric bootstrap and the square distance of Mahalanobis to verify which copulas best fit the data. We then see that the constant T-student and the time-varying Normal copulas have the best fit to the data regardless its origin. The copulas with asymmetric dependence structures are chosen only when the data origin is also a copula with asymmetric dependence.

Additionally, we evaluate the impact of different copulas on the Value at Risk (VaR) estimation. To do this, we calculate the VaR from simulations of the conditional joint distribution derived from the fitted copulas. All the time-varying conditional copulas show a good performance according to the tests of Kupiec (1995) and Christoffersen (1998), which may indicate that, when dealing with time-varying copulas, the copula functional form does not have much relevance in the VaR estimation. With these tests, we also believe that the SJC and RGC copulas, at the significance level of 1%, show more aggressive estimates, as they indicate more extreme quantiles than those expected for a particular level $\alpha$, while the Normal and T copulas provide more conservative estimates, i.e., estimated quantiles are less extreme than the VaR for fixed level (except for the pair IBOV/SP500, where all estimates, at 1% and 5%, are aggressive).

\textsuperscript{25} We consider as aggressive the VaR estimates whose estimated quantile is more extreme than the theoretical VaR, otherwise VaR is considered as conservative.
References


